MTL-algebras via rotations of basic hoops

Sara Ugolini

University of Denver, Department of Mathematics

(Ongoing joint work with P. Aglianò)

4th SYSMICS Workshop - September 16th 2018

A commutative, integral residuated lattice, or CIRL, is a structure $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 1)$ where:

(i) $(A, \wedge, \vee, 1)$ is a lattice with top element 1,

(ii) $(A, \cdot, 1)$ is a commutative monoid,

(iii) $(\cdot, \to)$ is a *residuated pair*, i.e. it holds for every $x, y, z \in A$:

$$x \cdot z \leq y \quad \text{iff} \quad z \leq x \to y.$$

CIRLs constitute a variety, $\mathbb{RL}$.

Examples: $(\mathbb{Z}^-, +, \ominus, min, max, 0)$, ideals of a commutative ring...

A bounded CIRL, or BCIRL, is a CIRL $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 0, 1)$ with an extra constant 0 that is the least element of the lattice.

Examples: Boolean algebras, Heyting algebras...

In every BCIRL we can define further operations and abbreviations:

$$\neg x = x \to 0, \quad x + y = \neg(\neg x \cdot \neg y), \quad x^2 = x \cdot x.$$

Totally ordered structures are called *chains*.

A bounded CIRL, or BCIRL, is a CIRL $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 0, 1)$ with an extra constant 0 that is the least element of the lattice.

Examples: Boolean algebras, Heyting algebras...

In every BCIRL we can define further operations and abbreviations:

$$\neg x = x \to 0, \quad x + y = \neg(\neg x \cdot \neg y), \quad x^2 = x \cdot x.$$

Totally ordered structures are called *chains*.

A CIRL, or BCIRL, is semilinear (or *prelinear*, or *representable*) if it is a subdirect product of chains.

We call semilinear CIRLs GMTL-algebras and semilinear BCIRLs MTL-algebras. They constitute varieties that we denote with $\mathbb{GMTL}$ and $\mathbb{MTL}$.

A bounded CIRL, or BCIRL, is a CIRL $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 0, 1)$ with an extra constant 0 that is the least element of the lattice.

Examples: Boolean algebras, Heyting algebras...

In every BCIRL we can define further operations and abbreviations:

$$\neg x = x \to 0, \quad x + y = \neg(\neg x \cdot \neg y), \quad x^2 = x \cdot x.$$

Totally ordered structures are called *chains*.

A CIRL, or BCIRL, is semilinear (or *prelinear*, or *representable*) if it is a subdirect product of chains.

We call semilinear CIRLs GMTL-algebras and semilinear BCIRLs MTL-algebras. They constitute varieties that we denote with $\mathbb{GMTL}$ and $\mathbb{MTL}$.

MTL-algebras are the semantics of Esteva and Godo's MTL, the fuzzy logic of left-continuous t-norms.

BL-algebras (semantics of Hàjek Basic Logic) are MTL-algebras satisfying divisibility: $x \wedge y = x \cdot (x \to y)$.

0-free reducts of BL-algebras (divisible GMTL-algebras) are known as basic hoops.

MV-algebras (semantics of Łukasiewicz logic) are involutive BL-algebras, i.e. they satisfy $\neg\neg x = x$.

0-free reducts of MV-algebras are called Wajsberg hoops.

Aglianò and Montagna in 2003, prove the following powerful characterization:

Theorem

Every totally ordered basic hoop (or BL algebra) is the ordinal sum of a family of Wajsberg hoops (whose first component is bounded).

Aglianò and Montagna in 2003, prove the following powerful characterization:

Theorem

Every totally ordered basic hoop (or BL algebra) is the ordinal sum of a family of Wajsberg hoops (whose first component is bounded).

Using the characterization in ordinal sums, Aglianò has been recently able to describe the splitting algebras in the variety of BL-algebras, and in relevant subvarieties, also providing the splitting equation.

Theorem

A BL-algebra is splitting in the lattice of subvarieties of $\mathbb{BL}$ if and only if it is a finite ordinal sum of Wajsberg hoops whose last component is isomorphic with the two elements Boolean algebra $\mathbf{2}$.

If $\mathbf{L}$ is any lattice a pair (a, b) of elements of L is a splitting pair if L is equal to the disjoint union of the ideal generated by a and the filter generated by b.

If $\mathbf{L}$ is any lattice a pair (a, b) of elements of L is a splitting pair if L is equal to the disjoint union of the ideal generated by a and the filter generated by b.

If $\mathbb{V}$ is any variety, we say that an algebra $\mathbf{A} \in \mathbb{V}$ is splitting in $\mathbb{V}$ if $\mathcal{V}(\mathbf{A})$ is the right member of a splitting pair in the lattice of subvarieties of $\mathbb{V}$.

Equivalently: $\mathbf{A}$ is splitting in $\mathbb{V}$ if there is a subvariety $\mathbb{W}_{\mathbf{A}} \subseteq \mathbb{V}$ (the conjugate variety of $\mathbf{A}$) such that for any variety $\mathbb{U} \subseteq \mathbb{V}$ either $\mathbb{U} \subseteq \mathbb{W}_{\mathbf{A}}$ or $\mathbf{A} \in \mathbb{U}$.

Some facts:

1. if $\mathbf{A}$ is splitting in $\mathbb{V}$ then $\mathbb{W}_{\mathbf{A}}^{\mathbb{V}}$ is axiomatized by a a single equation;
2. if $\mathbf{A}$ is splitting in $\mathbb{V}$ then $\mathbf{V}(\mathbf{A})$ is generated by a finitely generated subdirectly irreducible algebra;
3. if $\mathbf{A}$ is splitting in $\mathbb{V}$ then it is splitting in any subvariety of $\mathbb{V}$ to which it belongs.
4. If $\mathbb{V}$ is congruence distributive and generated by its finite members (FMP), then every splitting algebra in $\mathbb{V}$ is finite and uniquely determined by the splitting pair.

Theorem (Jankov, 1963)

Every finite subdirectly irreducible Heyting algebra is splitting in the variety of Heyting algebras.

Theorem (Jankov, 1963)

Every finite subdirectly irreducible Heyting algebra is splitting in the variety of Heyting algebras.

Theorem (Kowalski-Ono, 2000)

The two-element Boolean algebra $\mathbf{2}$ *is the only splitting algebra in the lattice of subvarieties of BCIRLs.*

MTL

Montagna, Noguera and Horčìk in 2006 prove that also MTL-chains allow a maximal decomposition in terms of ordinal sums of GMTL-algebras.

However, it is not currently known how to characterize GMTL-algebras, or MTL-algebras, that are sum-irreducible (any involutive MTL-algebra is sum irreducible).

Via the generalized rotation construction (Busaniche, Marcos and U., 2018), we will use results from the theory of basic hoops to shed light on the hard problem of understanding splitting algebras in some wide classes of MTL-algebras.

Generalized rotation

Let $\mathbf{R} = (R, \cdot, \to, \wedge, \vee, 1)$ be a RL

Let $\delta : R \to R$ be a nucleus operator:
i.e. a closure operator such that

$\delta(x) \cdot \delta(y) \leq \delta(x \cdot y)$,

that also respects the lattice operations.

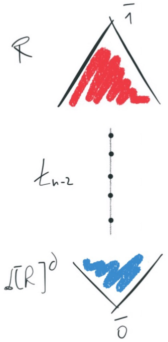
Examples: $id, \bar{1}$ ($\bar{1}(x) = 1$, for every $x \in R$).

Generalized rotation

Let $\mathbf{R} = (R, \cdot, \to, \wedge, \vee, 1)$ be a RL, δ wdl-admissible and $n \in \mathbb{N}, n \geq 2$. We define the generalized rotation $\mathfrak{R}_n^\delta(R)$:

Generalized rotation

Let $\mathbf{R} = (R, \cdot, \to, \wedge, \vee, 1)$ be a RL, δ wdl-admissible and $n \in \mathbb{N}, n \geq 2$. We define the generalized rotation $\mathfrak{R}_n^\delta(R)$:



We can see the domain of $\mathfrak{R}_n^\delta(R)$ as:

$$(\{1\}\times R) \quad \cup \quad \{\{s\}\times\{1\}\}_{s\in Ł_n\setminus\{0,1\}} \quad \cup \quad (\{0\}\times\delta[R])$$

Generalized rotation

Let $\mathbf{R} = (R, \cdot, \to, \wedge, \vee, 1)$ be a RL, δ wdl-admissible and $n \in \mathbb{N}, n \geq 2$. We define the generalized rotation $\mathfrak{R}_n^\delta(R)$:

We can see the domain of $\mathfrak{R}_n^\delta(R)$ as:

$$(\{1\}\times R) \quad \cup \quad \{\{s\}\times\{1\}\}_{s\in Ł_n\setminus\{0,1\}} \quad \cup \quad (\{0\}\times\delta[R])$$

With suitably defined operations, $\mathfrak{R}_n^\delta(R)$ is a directly indecomposable bounded RL (Busaniche, Marcos, U. 2018).

With $\delta = \bar{1}$, $\mathfrak{R}_n^{\bar{1}}(R)$ is the n-lifting of **R**:

Stonean residuated lattices (SMTL-algebras, Gödel algebras, product algebras), BL_n-algebras...

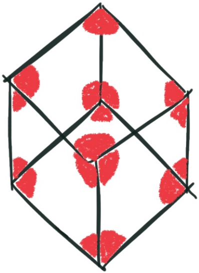
With $\delta = \bar{1}$, $\mathfrak{R}_n^{\bar{1}}(R)$ is the n-lifting of **R**:

Stonean residuated lattices (SMTL-algebras, Gödel algebras, product algebras), BL_n-algebras...

With $\delta = id$, $\mathfrak{R}_n^{id}(R)$ is the disconnected n-rotation of **R**:

Disconnected rotations (perfect MV-algebras, NM^-...), connected rotations (nilpotent minimum NM, regular Nelson lattices)

$n = 2$: srDL-algebras



(Cignoli and Torrens 2006, Aguzzoli, Flaminio and U. 2017): Generalized rotation with $n = 2$ generate srDL-algebras: MTL-algebras that satisfy:

$$(DL) \qquad (2x)^2 = 2x^2$$

$$(r) \qquad \neg(x^2) \to (\neg\neg x \to x) = 1$$

(Aguzzoli, Flaminio, U., 2017) srDL-algebras are equivalent to categories whose objects are **quadruples** $(\mathbf{B}, \mathbf{R}, \vee_e, \delta)$:

- $\mathbf{B}$ is a Boolean algebra,
- $\mathbf{R}$ is a GMTL-algebra,
- a δ operator,
- $\vee_e : B \times R \to R$ is an *external join*

Dualized construction

Let $\mathbf{A}$ be an srDL-algebra, and u, v, w... be the ultrafilters of its Boolean skeleton.

Dualized construction

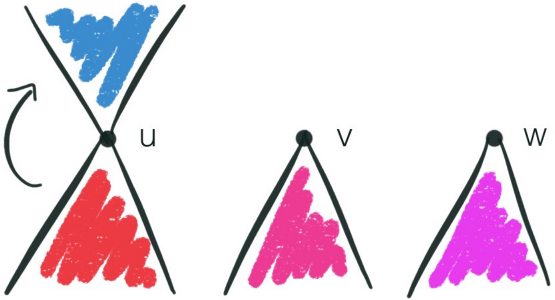
Below u: the prime lattice filters of the radical that respect an "external primality condition" wrt u, ordered by inclusion.

Dualized construction

Same for $v, w, \ldots$

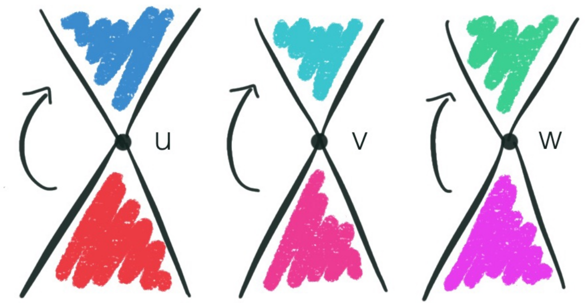
Dualized construction

Rotate upwards the δ-images of the elements below u.

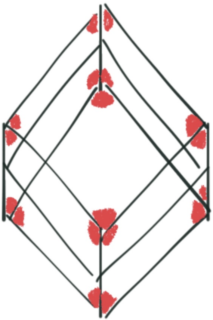


Dualized construction

The dualized rotation construction obtained is isomorphic to the poset of prime lattice filters of $\mathbf{A}$ (Fussner, Ugolini 2018).

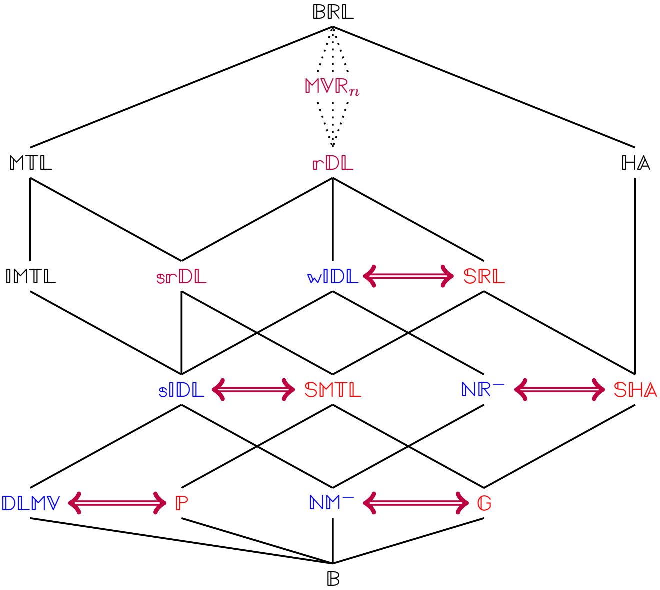


$n \geq 2$: MVR$_n$-algebras



MVR$_n$-algebras constitute a variety and have an MV-retraction term (thus, an MV-skeleton) and are equivalent to categories whose objects are **quadruples** $(\mathbf{M}, \mathbf{R}, \vee_e, \delta)$ where $\mathbf{M}$ is an MV$_n$-algebra.

Via these categorical characterization, we prove that the full subcategories of MVR$_n$-algebras generated by, respectively, n-liftings and generalized disconnected rotations are categorically equivalent.


BRL
MVR$_n$
MTL
rDL
HA
IMTL
srDL
wIDL
SRL
sIDL
SMTL
NR$^-$
SHA
DLMV
P
NM$^-$
G
B

Let us consider a δ-operator that is term-defined (briefly, td-rotation), i.e. $\delta(x)$ is a unary term (e.g. $\delta(x) = x$ and $\delta(x) = 1$).

Let $\mathbb{K}$ be any class of GMTL-algebras and let δ be a td-rotation; for **A** in $\mathbb{K}$ we denote by $\mathbf{A}^{\delta_n}$ its generalized n-rotation and we define

$$\mathbb{K}^{\delta_n} = \{\mathbf{A}^{\delta_m} : m-1 \mid n-1, \mathbf{A} \in \mathbb{K}\}.$$

Let us consider a δ-operator that is term-defined (briefly, td-rotation), i.e. $\delta(x)$ is a unary term (e.g. $\delta(x) = x$ and $\delta(x) = 1$).

Let $\mathbb{K}$ be any class of GMTL-algebras and let δ be a td-rotation; for $\mathbf{A}$ in $\mathbb{K}$ we denote by $\mathbf{A}^{\delta_n}$ its generalized n-rotation and we define

$$\mathbb{K}^{\delta_n} = \{\mathbf{A}^{\delta_m} : m-1 \mid n-1, \mathbf{A} \in \mathbb{K}\}.$$

From now on we will write δ for δ_2.

Lemma

Let $\mathbb{K}$ and δ as above; then

1. $\boldsymbol{H}(\mathbb{K})^{\delta_n} = \boldsymbol{H}(\mathbb{K}^{\delta_n})$;
2. $\boldsymbol{S}(\mathbb{K})^{\delta_n} = \boldsymbol{S}(\mathbb{K}^{\delta_n})$;
3. $\boldsymbol{P}_u(\mathbb{K})^{\delta_n} \subseteq \boldsymbol{ISP}_u(\mathbb{K}^{\delta_n})$;
4. $\boldsymbol{P}_u(\mathbb{K}^{\delta}) \subseteq \boldsymbol{IS}(\boldsymbol{P}_u(\mathbb{K})^{\delta})$.

Corollary

Let $\mathbb{K}$ be a class of GMTL-algebras and δ a td-rotation; then $\mathbf{A} \in \boldsymbol{HSP}_u(\mathbb{K})$ if and only if $\mathbf{A}^\delta \in \boldsymbol{HSP}_u(\mathbb{K}^\delta)$. Moreover for any $n \geq 2$, $\mathbf{A} \in \boldsymbol{HSP}_u(\mathbb{K})$ implies $\mathbf{A}^{\delta_n} \in \boldsymbol{HSP}_u(\mathbb{K}^{\delta_n})$.

Corollary

Let $\mathbb{K}$ be a variety of GMTL-algebras and δ a td-rotation; then the mapping

$$\mathbb{V} \longmapsto \mathbb{V}^\delta$$

is an isomorphism between the lattice of subvarieties of $\mathbb{K}$ and the lattice of subvarieties of $\mathbb{K}_\delta$, where the inverse is

$$\mathbb{W} \longmapsto \mathbb{W}^{\mathscr{R}} = \boldsymbol{V}(\mathscr{R}(\mathbf{A}) : \mathbf{A} \in \mathbb{W}).$$

Lemma

Given $\mathbb{V}$ a variety of GMTL-algebras, $\mathbf{A}$ is directly indecomposable in $\mathbb{V}^{\delta_n}$ iff the radical $\mathscr{R}(\mathbf{A}) \in \mathbb{V}$ and the MV-skeleton $\mathscr{M}(\mathbf{A}) \cong Ł_m$ for $m \in \mathbb{N}$ such that $m - 1 \mid n - 1$.

Proposition

Given $\mathbb{V}, \mathbb{W}$ varieties of $GMTL$-algebras, $\mathbb{V}^{\delta_m} \subseteq \mathbb{W}^{\delta_n}$ iff $m - 1 \mid n - 1$ and $\mathbb{V} \subseteq \mathbb{W}$.

Theorem

For any variety $\mathbb{V}$ of GMTL-algebras an algebra $\mathbf{A}$ is splitting in $\mathbb{V}$ if and only if $\mathbf{A}^\delta$ is splitting in $\mathbb{V}^\delta$ if and only if $\mathbf{A}^\delta$ is splitting in $\mathbb{V}^{\delta_n}$.

Theorem

For any variety $\mathbb{V}$ of GMTL-algebras an algebra $\mathbf{A}$ *is splitting in* $\mathbb{V}$ *if and only if* $\mathbf{A}^{\delta}$ *is splitting in* $\mathbb{V}^{\delta}$ *if and only if* $\mathbf{A}^{\delta}$ *is splitting in* $\mathbb{V}^{\delta_n}$*.*

There is more: e.g., since $\boldsymbol{V}(\text{Ł}_2)$ is the only atom in any $\mathbb{V}^{\delta_n}$, it is splitting with conjugate variety the trivial variety. Moreover:

Theorem

For any variety $\mathbb{V}$ of GMTL-algebras an algebra $\mathbf{A}$ *is splitting in* $\mathbb{V}$ *if and only if* $\mathbf{A}^{\delta}$ *is splitting in* $\mathbb{V}^{\delta}$ *if and only if* $\mathbf{A}^{\delta}$ *is splitting in* $\mathbb{V}^{\delta_n}$*.*

There is more: e.g., since $\boldsymbol{V}(\mathbf{Ł}_2)$ is the only atom in any $\mathbb{V}^{\delta_n}$, it is splitting with conjugate variety the trivial variety. Moreover:

Lemma

Suppose $\mathbb{V}$ is a variety of GMTL-algebras and suppose that n is such that $n-1$ is a prime power; $\mathbf{Ł}_n$ *is splitting in* $\mathbb{V}^{\delta_n}$*.*

Lemma

Let $\mathbb{V}$ be a variety of GMTL-algebras that is completely join irreducible in the lattice of subvarieties of $\mathbb{V}$ and let $\mathbf{A}$ *such that* $\boldsymbol{V}(\mathbf{A}) = \mathbb{V}$*. Then* $\mathbf{A}^{\delta_m}$ *is splitting in* $\mathbb{V}^{\delta_n}$ *for any m such that $m-1 \mid n-1$.*

Theorem

For any variety $\mathbb{V}$ of GMTL-algebras an algebra $\mathbf{A}$ is splitting in $\mathbb{V}$ if and only if $\mathbf{A}^{\delta}$ is splitting in $\mathbb{V}^{\delta}$ if and only if $\mathbf{A}^{\delta}$ is splitting in $\mathbb{V}^{\delta_n}$.

There is more: e.g., since $\boldsymbol{V}(\mathbf{Ł}_2)$ is the only atom in any $\mathbb{V}^{\delta_n}$, it is splitting with conjugate variety the trivial variety. Moreover:

Lemma

Suppose $\mathbb{V}$ is a variety of GMTL-algebras and suppose that n is such that $n-1$ is a prime power; $\mathbf{Ł}_n$ is splitting in $\mathbb{V}^{\delta_n}$.

Lemma

Let $\mathbb{V}$ be a variety of GMTL-algebras that is completely join irreducible in the lattice of subvarieties of $\mathbb{V}$ and let $\mathbf{A}$ such that $\boldsymbol{V}(\mathbf{A}) = \mathbb{V}$. Then $\mathbf{A}^{\delta_m}$ is splitting in $\mathbb{V}^{\delta_n}$ for any m such that $m-1 \mid n-1$.

Problem: finding splittings in $\mathbb{GMTL}$ is not easier than finding them in $\mathbb{MTL}$. Thus we are going to use our construction to transfer results from $\mathbb{BH}$ to $\mathbb{MTL}$.

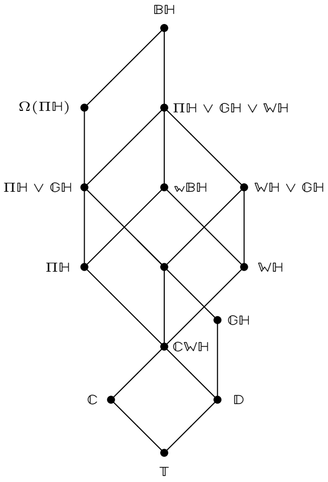


Figure: The lattice of subvarieties of $\mathbb{BH}$

Disconnected n-rotations of basic hoops

The n-liftings of basic hoops are BL-algebras, so we are interested in the disconnected n-rotations.

Disconnected n-rotations of basic hoops

The n-liftings of basic hoops are BL-algebras, so we are interested in the disconnected n-rotations.

In general the variety $\mathbb{BH}_n^\delta$ of involutive MTL-algebras generated by all the n-rotations of basic hoops can be axiomatized by

$$(\nabla_n(x) \wedge \nabla_n(y)) \to ((x(x \to y)) \to (y(y \to x))) \approx 1.$$

The k-rotations of splitting algebras in $\mathbb{BH}$, whenever $k-1 \mid n-1$, are splitting algebras in $\mathbb{BH}_n^\delta$.

If $\mathbf{A}$ is splitting in $\mathbb{BH}$ with splitting equation $\tau(x_1, \ldots, x_k) \approx 1$ then the splitting equation of $\mathbf{A}_2^\delta$ in $\mathbb{BH}_n^\delta$ is

$$\bigwedge_{i=1}^{k} \nabla_2(x_i) \to \tau(x_1, \ldots, x_k) \approx 1.$$

Disconnected n-rotations of basic hoops

Cancellative hoops

Proposition

Given a GMTL-algebra $\mathbf{R}$, its disconnected n-rotation is a BL-algebra if and only if $n = 2$ and $\mathbf{R}$ is a cancellative hoop.

We will refer to the varieties $\mathbb{C}^{\delta_n}$ for $n \geq 2$ as **nilpotent product varieties**. Since cancellative hoops are axiomatized relative to Wajsberg hoop by $(x \to x^2) \to x \approx 1$ the variety $\mathbb{C}^{\delta_n}$ are axiomatized by

$$\neg x^n \to ((x \to x^2) \to x) \approx 1.$$

Disconnected n-rotations of basic hoops

Cancellative hoops

Proposition

Given a GMTL-algebra $\mathbf{R}$, its disconnected n-rotation is a BL-algebra if and only if $n = 2$ and $\mathbf{R}$ is a cancellative hoop.

We will refer to the varieties $\mathbb{C}^{\delta_n}$ for $n \geq 2$ as **nilpotent product varieties**. Since cancellative hoops are axiomatized relative to Wajsberg hoop by $(x \to x^2) \to x \approx 1$ the variety $\mathbb{C}^{\delta_n}$ are axiomatized by

$$\neg x^n \to ((x \to x^2) \to x) \approx 1.$$

Lemma

$\mathbb{C}^{\delta_n} = \boldsymbol{V}(\mathbf{C}_\omega^{\delta_n})$.

Thus e.g. the lattice of subvarieties of $\mathbb{C}^{\delta_3}$ is the three element chain where the only proper nontrivial subvariety is $\mathbb{C}^{\delta_2}$, Chang MV-algebra.

Disconnected n-rotations of basic hoops

Wajsberg hoops

The variety $\mathbb{BH}_n^\delta$ generated by all the n-rotations of Wajsberg hoops can be axiomatized by

$$(\nabla_n(x) \wedge \nabla_n(y)) \to (((x \to y) \to y) \to ((y \to x) \to x)) \approx 1.$$

The only splitting algebras in $\mathbb{WH}$ are $\mathbf{C}_\omega$ and $\mathbf{Ł}_2$, while proper subvarieties of $\mathbb{WH}$ are all generated by finitely many finite chains [Agliano, Panti 1999], so their lattice of subvarieties is finite.

Disconnected n-rotations of basic hoops

Wajsberg hoops

The variety $\mathbb{BH}_n^\delta$ generated by all the n-rotations of Wajsberg hoops can be axiomatized by

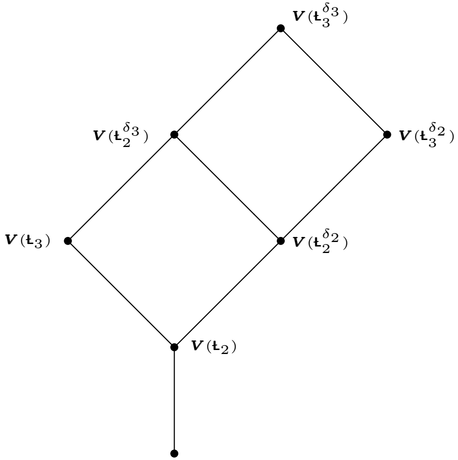
$$(\nabla_n(x) \wedge \nabla_n(y)) \to (((x \to y) \to y) \to ((y \to x) \to x)) \approx 1.$$

The only splitting algebras in $\mathbb{WH}$ are $\mathbf{C}_\omega$ and $\mathbf{Ł}_2$, while proper subvarieties of $\mathbb{WH}$ are all generated by finitely many finite chains [Agliano, Panti 1999], so their lattice of subvarieties is finite.

Thus the splitting algebras in a proper subvariety $\mathbb{V}$ of $\mathbb{WH}$ are exactly the totally ordered ones that generate a proper variety that is join irreducible in the lattice of subvarieties of $\mathbb{V}$.

We will refer to the varieties $\mathbb{V}^{\delta_n}$ for $n \geq 2$ as **nilpotent Łukasiewicz varieties**.

Any proper variety of Wajsberg hoops is axiomatized (modulo basic hoops) by a single equation in one variable of the form $t_{\mathbb{V}}(x) \approx 1$. Thus $\mathbb{V}^{\delta_n}$ is axiomatized by $\neg x^n \vee t_{\mathbb{V}}(x) \approx 1$.



Let us consider $\boldsymbol{V}(\mathbf{Ł}_3^{\delta_3})$. The splitting algebras are $\mathbf{Ł}_2$, $\mathbf{Ł}_2^{\delta_2}$, $\mathbf{Ł}_3$ and $\mathbf{Ł}_3^{\delta_2}$.
For greater n we get more complex lattices of subvarieties and more splitting algebras.

Disconnected n-rotations of basic hoops

Gödel hoops

Let $\mathbf{G}_n$ be the Gödel chain with n-elements; then (Aglianò 2017) each $\mathbf{G}_n$ is splitting in $\mathbb{GH}$ with splitting equation $\bigwedge_{i=0}^{n-1}((x_{i+1} \to x_i) \to x_i \leq \bigvee_{i=0}^{n} x_i$.

We can axiomatize $\mathbb{NM} = \mathbb{GH}^{\delta_3}$ relatively to involutive MTL-algebras as

$$\neg x^2 \vee (x^2 \to x) \approx 1.$$

Similarly, $\mathbb{GH}^{\delta_4}$ is axiomatized relatively to involutive MTL-algebras by

$$\neg x^3 \vee (x \to x^2) \approx 1$$

and the splitting algebras in $\mathbb{NM}_4$ are exactly $\mathbf{G}_l^{\delta}$ for $l \geq 2$.

Disconnected n-rotations of basic hoops

Gödel hoops

Let $\mathbf{G}_n$ be the Gödel chain with n-elements; then (Aglianò 2017) each $\mathbf{G}_n$ is splitting in $\mathbb{GH}$ with splitting equation $\bigwedge_{i=0}^{n-1}((x_{i+1} \to x_i) \to x_i \leq \bigvee_{i=0}^{n} x_i$.

We can axiomatize $\mathbb{NM} = \mathbb{GH}^{\delta_3}$ relatively to involutive MTL-algebras as

$$\neg x^2 \vee (x^2 \to x) \approx 1.$$

Similarly, $\mathbb{GH}^{\delta_4}$ is axiomatized relatively to involutive MTL-algebras by

$$\neg x^3 \vee (x \to x^2) \approx 1$$

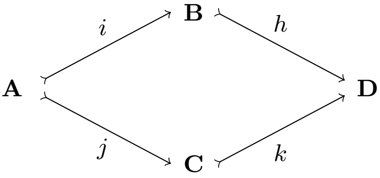
and the splitting algebras in $\mathbb{NM}_4$ are exactly $\mathbf{G}_l^{\delta}$ for $l \geq 2$.

The lattice of subvarieties of $\mathbb{NM}_4$ is identical to the lattice of subvarieties of $\mathbb{NM}$ ($3-1$ and $4-1$ have the same number of divisors and they are all relatively prime).

The case $n=5$ is similar but harder, however the lattice of subvarieties should still be understandable as a "higher dimensional" version of the one of NM-varieties (ongoing work).

Amalgamation

Let $\mathbb{K}$ be a class of algebras of the same type. We say that $\mathbb{K}$ has the amalgamation property (AP for short) iff whenever $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are in $\mathbb{K}$, and i and j are monomorphisms from $\mathbf{A}$ into $\mathbf{B}$ and from $\mathbf{A}$ into $\mathbf{C}$ respectively, there are $\mathbf{D} \in \mathbb{K}$ and monomorphisms h and k from $\mathbf{B}$ into $\mathbf{D}$ and from $\mathbf{C}$ into $\mathbf{D}$ such that the compositions $h \circ i$ and $k \circ j$ coincide. In this case, $(\mathbf{D}, h, k)$ is said to be an *amalgam* of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$.



Metcalfe, Montagna and Tsinakis show that a variety $\mathbb{V}$ of semilinear (not necessarily commutative and integral) residuated lattices satisfying the congruence extension property has the AP iff the class of chains in $\mathbb{V}$ has AP.

Theorem

A variety $\mathbb{V}$ of GMTL-algebras has the AP iff $\mathbb{V}^{\delta_n}$ has the AP.

Since (Montagna 2006) the varieties of basic hoops $\mathbb{BH}$, Wajsberg hoops $\mathbb{WH}$, cancellative hoops $\mathbb{CH}$ and Gödel hoops $\mathbb{GH}$ have the AP, each one of $\mathbb{BH}^{\delta_n}$, $\mathbb{WH}^{\delta_n}$, $\mathbb{CH}^{\delta_n}$ and $\mathbb{GH}^{\delta_n}$ has AP, thus in particular:

- the variety generated by perfect MV-algebras (Di Nola, Lettieri 1994) and all nilpotent product varieties;
- NM and NM^{-} (Bianchi 2001), and all nilpotent minimum varieties;
- all nilpotent Łukasiewicz varieties.

Thank you.