Greatest Hits of the Hawaiian Legends of Universal Algebra and Lattice Theory

Peter Jipsen

Chapman University

ALH 2018: Conference on Algebra and Lattice Theory
Honoring Ralph Freese, Bill Lampe, and JB Nation

June 24, 2018

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Exception: correct mathematical results can be exported from these islands

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But also include a few overlooked gems

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JB and Ralph were still working on their 10000 hours

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JB showed in his undergraduate thesis (among other results) that any such group must have in its subgroup lattice a countable sublattice of height 2

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Platinum: ms ≥ 10 or gs ≥ 30

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Bill proves that you can pick any group G and any pair of nontrivial algebraic lattices L, L' and find an algebra A such that

$$\mathrm{Aut}(A) \cong G,\ \mathrm{Sub}(A) \cong L \text{ and } \mathrm{Con}(A) \cong L'$$

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Representations of lattices as congruence lattices (16 pages)

Greatest Hits from the mid 1970s

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1974 Ralph, DM (ms6, gs42)

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An application of Dilworth's lattice of maximal antichains

Dilworth proved that the maximal antichains of a poset form a distributive lattice.

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Ideal lattices of lattices
Every algebraic lattice is a subdirect product of complete upper continuous subdirectly irreducible lattices

More Greatest Hits from the mid 1970s

Planar sublattices of FM(4)

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"In this paper we show that any finite planar modular lattice which does not contain a sublattice isomorphic to M_4 is a sublattice of $FM(4)$. [...] infinitely many of these lattices are simple, whereas previously no example of a simple sublattice of FM(4) with more than five elements was known."

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"Theorem: Every lattice variety is generated by its strongly atomic members.

Every modular lattice variety is generated by its weakly atomic, subdirectly irreducible members.

Every modular lattice can be imbedded in a strongly atomic, subdirectly irreducible modular lattice"

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Projective lattices

1978 JB & Ralph, PJM (ms7, gs44)
Necessary and sufficient conditions for a lattice to be projective (= retract

More Greatest Hits from the late 1970s

The Legendary UA-Team

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Projective geometries as projective modular lattices

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Using von Neumann's n-frames to coordinatize modular lattices, it is shown that the lattice of subspaces of a finite ($n \geq 4$) dimensional vector space over a finite prime field is projective in the class of modular lattices. It follows that $L(\mathbb{Z}_p^n)$ is a splitting modular lattice (as well as some gluings of these).

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The variety of modular lattices is not generated by its finite members
"This paper proves the result of the title." (there exists a 5 variable identity)
Every free distributive lattice can be embedded into a free modular lattice.
Epimorphisms of modular lattices are not necessarily onto.
Is every splitting modular lattice finite?

More Greatest Hits from the late 1970s

Congruence lattices of algebras of fixed similarity type. I

More Greatest Hits from the late 1970s

Congruence lattices of algebras of fixed similarity type. I

1979 Bill, Ralph & Walter, PJM (ms8, gs56)
"We prove that if V is any infinite dimensional vector space over any uncountable field F, then the congruence lattice (= subspace lattice) of V cannot be represented as a congruence lattice (of any algebra) without using at least $|F|$ operations. This refutes a long standing conjecture—that one binary operation would always suffice."

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Uses von Neumann n-frames to interpret the unsolvability of the word problem for groups.

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A characterization of identities implying congruence modularity. I

1980 Alan & Ralph, CJM (ms11, gs74)

From Ralph's 1995 paper on Alan's early work: "After completing [9], Bjarni Jónsson and J. B. Nation accused Alan and I of killing the subject of congruence varieties, since this paper answered most of the questions we had been interested in. [...] However, the subject is coming back to life because of some important recent developments. There are also some very interesting unsolved problems. One problem that is still open: Is there a unique largest modular congruence variety?"

Greatest Hits from the early 1980s

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Residually small varieties with modular congruence lattices

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From Walter Taylor's review: "The first main result [...] says that if V is [cong. modular and] residually small, then $\alpha \wedge [\beta, \beta] \leq [\alpha, \beta]$ for all α, $\beta \in A \in V$. In fact, if this commutator identity fails, then residual smallness fails quite sharply: there exist subdirectly irreducible algebras in V with arbitrarily large congruence lattices. The proof is a **straightforward but very clever** calculation in lattice theory.

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1981 András, Christian & Ralph, AU (ms6, gs31)

More Greatest Hits from the early 1980s

The Ghostbusters

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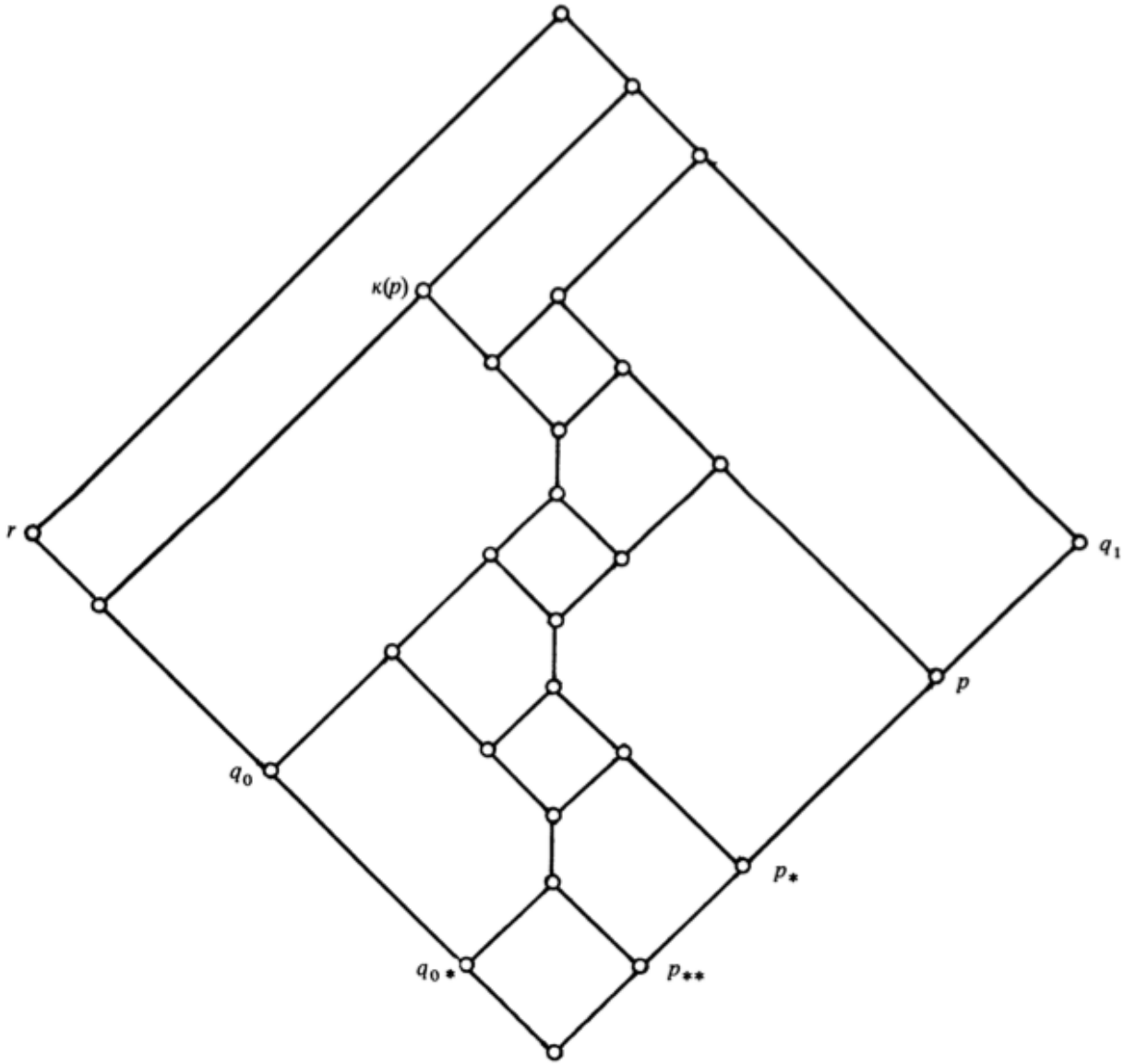
In which paper did this lattice appear?



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1982 JB, TAMS (ms4, gs31)

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Finite sublattices of a free lattice

"Every finite semidistributive lattice satisfying Whitman's condition is isomorphic to a sublattice of a free lattice."

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Covers in free lattices

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Commutator theory for congruence modular varieties

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Greatest Hits from the late 1980s

The Legendary Musketeers

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Greatest Hits from the late 1980s

Arguesian lattices whose skeleton is a chain

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Finitely presented lattices: canonical forms and the covering relation

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Greatest Hits from the early 1990s

An approach to lattice varieties of finite height

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Among many other results, lattice identities $\sigma_n, \tau_{n,k}, \lambda_m$ are defined such that a subdirectly irreducible lattice is finite if and only if it satisfies these identities for some $k, m, n \geq 1$. The aim is to solve the finite height conjecture.

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1994 Alan, Bjarni, Christian, Doug & JB, AU (ms3, gs11)

Greatest Hits from the mid 1990s

The Legendary Tenors

Greatest Hits from the mid 1990s

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A perspective on algebraic representations of lattices

Greatest Hits from the mid 1990s

A perspective on algebraic representations of lattices

1994 Bill, AU (ms5, gs15)

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The three Legends taking off

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The three Legends taking off

Someone who did grow up on Hawaii

Jimmy Freese

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Greatest Hits of the Future

Greatest Hits of the Future

Ralph: “Hey, check out my UACalc app

Greatest Hits of the Future

Ralph: "Hey, check out my UACalc app
with AI machine learning on the iPhone!"

Greatest Hits of the Future

JB: “Wowza! I found a projective plane of order 12

Greatest Hits of the Future

JB: "Wowza! I found a projective plane of order 12 with my quantum computer!"

Greatest Hits of the Future

Bill: "Guys, I created an alternative universe where all finite lattices are represented by finite algebras;

Greatest Hits of the Future

Bill: "Guys, I created an alternative universe where all finite lattices are represented by finite algebras; it's easy"

Greatest Hits of the Future

Live long and prosper!

Greatest Hits of the Future

Live long and prosper!
Thank you Bill, JB and Ralph!

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Thank you Bill, JB and Ralph!