The structure of idempotent involutive residuated lattices and weakening relation algebras

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Overview

Part I, with O. Tuyt and D. Valota

- Commutative idempotent involutive residuated lattices
- Gluing construction
- Ungluing decomposition

Part II, with N. Galatos

- $\mathbf{FL}_2$-algebras and their congruences
- Weakening relation algebras
- Double-division conuclei
Involutive residuated lattices

**Definition**

A **pointed residuated lattice** $A = \langle A, \land, \lor, \cdot, \backslash, /, 1, 0 \rangle$ is

- a lattice $\langle A, \land, \lor \rangle$ and a monoid $\langle A, \cdot, 1 \rangle$ such that

  $$x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z$$

  for all $x, y, z \in A$.

$A$ is **involutive** if $\sim -x = x = -\sim x$, where $\sim x = x\backslash 0$ and $-x = 0/x$.

\backslash, / can be term-defined via $x\backslash y = \sim (-y \cdot x)$ and $x/y = -(y \cdot \sim x)$.

- $A$ is **commutative** if $x \cdot y = y \cdot x$ (hence $-x = \sim x$)
- $A$ is **idempotent** if $x \cdot x = x$ for all $x \in A$

$CldInRL$ denotes the variety of **commutative idempotent involutive residuated lattices**.
Examples of ClIdINRLs

Let $A \in \text{ClIdInRL}$.

- $\langle A, \cdot, 1 \rangle$ is a meet-semilattice with top element 1 and order $\sqsubseteq$ (monoidal order) defined as

  $$a \sqsubseteq b \iff a \cdot b = a.$$  

Hence, the orders $\leq$ and $\sqsubseteq$ together with the involution $\neg$ completely determine $A$, allowing us to work in the signature $\langle A, \lor, \cdot, \neg, 0, 1 \rangle$

- **Boolean algebras** (where $\leq = \sqsubseteq$)

- **Sugihara monoids** defined as distributive ClIdInRLs (=$\text{algebraic semantics for relevance logic RM}^t$)

  Dunn [1970] proved that the subdirectly irreducible Sugihara monoids are linearly ordered. Up to isomorphism, there is one such algebra $S_n$ for each chain with $n$ elements.
Another example

\[ \langle A, \leq \rangle \]

\[ \langle A, \sqsubseteq \rangle \]
Another example

\[ \langle A, \leq \rangle \]

\[ \langle A, \sqsubseteq \rangle \]
Another example

\[ \langle A, \leq \rangle \]

\[ \langle A, \sqsubseteq \rangle \]
Some properties

For each $x \in A$, let

- $0_x := x \land -x = x \cdot -x$
- $1_x := x \lor -x = -(x \cdot -x) = x/x$
- $\mathbb{B}_x := \{y \in A \mid 0_x \sqsubseteq y \sqsubseteq 1_x\}$
- $\downarrow 0 := \{y \in A \mid y \leq 0\} = \{0_x \mid x \in A\}$

**Lemma**

- For each $x \in A$, $\langle \mathbb{B}_x, \land, \lor, -, 0_x, 1_x \rangle$ is a **Boolean algebra**
- For each $x \in A$, the monoidal order and the lattice order agree on $\mathbb{B}_x$
- The monoidal intervals $\mathbb{B}_x$ **partition** $A$
- $\langle \downarrow 0, \cdot, \lor \rangle$ is a **distributive lattice** with top element $0$

Hence, the monoidal semilattice is a disjoint union of Boolean algebras over the ‘skeleton’ of a distributive lattice.
Construction: example of $\mathbf{C} = \mathbf{A} \oplus \phi \mathbf{B}$
Construction: formally

Let $\uparrow a = \{ x \in A \mid a \subseteq x \}$ and $\downarrow b = \{ x \in B \mid x \subseteq b \}$.  

$A = \langle A, \lor^A, \cdot^A, \land^A, 0^A, 1^A \rangle$ (the bottom algebra) and $B = \langle B, \lor^B, \cdot^B, \land^B, 0^B, 1^B \rangle$ (the top algebra) are $\varphi$-compatible if

- $\varphi$ is a bijection $\uparrow a \to \downarrow b$ for some $a \leq 1^A$ and $0^B \leq b \leq 1^B$ such that
- $\varphi$ preserves join, i.e. $\varphi(x \lor^A y) = \varphi(x) \lor^B \varphi(y)$
- $\varphi$ preserves fusion, i.e. $\varphi(x \cdot^A y) = \varphi(x) \cdot^B \varphi(y)$ and
- $0^B = \varphi(a \lor^A 0^A)$.

For $\varphi$-compatible algebras we define a glueing construction $\oplus \varphi$. 

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Glueing construction

\[ A \oplus_\varphi B := \langle A \cup B, \lor, \cdot, -, 1^B, 0^B \rangle \]

\[
x \lor y = \begin{cases} 
x \lor^A y & \text{if } x, y \in A \\
x \lor^B y & \text{if } x, y \in B \\
\varphi(x \lor^A a) \lor^B y & \text{if } x \in A, y \in B, x \leq^A -^A a \\
x \lor^A \varphi^{-1}(y \cdot^B b) & \text{if } x \in A, y \in B, x \not\leq^A -^A a 
\end{cases}
\]

\[
x \cdot y = \begin{cases} 
x \cdot^A y & \text{if } x, y \in A \\
x \cdot^B y & \text{if } x, y \in B \\
x \cdot^A \varphi^{-1}(y \cdot^B b) & \text{if } x \in A, y \in B 
\end{cases}
\]

\[
-x = \begin{cases} 
-A x & \text{if } x \in A \\
-B x & \text{if } x \in B 
\end{cases}
\]
Theorem

For \( \varphi \)-compatible \( A, B \in \text{CldInRL} \) the algebra \( A \oplus_\varphi B \) is in \( \text{CldInRL} \).

The proof is by case analysis and direct computation.
For finite $C \in \text{CIdInRL}$, consider a co-atom $c$ in the underlying distributive lattice with universe $\downarrow 0 = \{0_x \mid x \in C\}$.

By distributivity, there exists $c^*$ such that $\langle c, c^* \rangle$ is a splitting pair of $\downarrow 0$.

Note: $c = 0_c$, hence $-c = 1_c$.

**Lemma**

The pair $\langle 1_c, c^* \rangle$ is a splitting pair of $(C, \sqsubseteq)$.

Moreover, $\uparrow c^*$ is a subuniverse of $C$, and $\downarrow 1_c$ is closed under $\lor, \cdot, -$.
Let $A = \langle \bot 1_c, \lor, \cdot, -, 1_c, 0_c \rangle$.

Let $B$ be the subalgebra of $C$ with subuniverse $\uparrow c^*$.

Choose $a = 1_c \cdot c^*$ and $b = (1_c \lor -a) \lor c^*$, and define

$\varphi(x) = (x \land -a) \lor c^*$ for $a \sqsubseteq x \sqsubseteq 1_c$.

**Lemma**

- $a \leq 1_c$ and $0 \leq b \leq 1$
- $\varphi$ is a bijection to $\{ y \mid c^* \sqsubseteq y \sqsubseteq b \}$ with $\varphi^{-1}(y) = y \cdot 1_c$
- $\varphi(c \lor a) = 0_b$

**Theorem**

*The algebra $C \in \text{CldInRL}$ is isomorphic to $A \oplus \varphi B$.*
The discovery of the previous theorem and the results below were guided by Prover9/Mace4 computations of all ClldInRLs with \( \leq 16 \) elements.

**Theorem**

Any finite member \( A \) of ClldInRL can be constructed using the gluing construction, starting from finite Boolean algebras.

**Corollary**

Any finite \( A \in ClldInRL \) is determined by its fusion semilattice and also by its lattice reduct.

To do: Implement an algorithm for constructing all finite ClldInRLs.
As an application, call an \( A \in \text{CIdInRL} \) fusion-distributive if the meet-semilattice \( \langle A, \cdot \rangle \) is distributive, i.e. if for all \( x, y, z \in A \),

\[
x \cdot y \sqsubseteq z \implies \exists x', y' \in A \text{ such that } x \sqsubseteq x', y \sqsubseteq y', \text{ and } z = x' \cdot y'.
\]

**Lemma**

For compatible fusion-distributive \( A, B \in \text{CIdInRL} \), their gluing \( C \) is fusion-distributive.

**Corollary**

- Any finite \( A \in \text{CIdInRL} \) is fusion-distributive.
- Every finite distributive lattice can occur as skeleton.
A one-generated infinite CIdInRL
A one-generated infinite ClfInRL
A one-generated infinite CI\dlnRL
A one-generated infinite ClIdlnRL
A one-generated infinite CIdInRL
A one-generated infinite CIdInRL
A one-generated infinite CIdInRL
A one-generated infinite ClDInRL
A one-generated infinite CIdInRL
A one-generated infinite ClDInRL

\[ x \wedge 1 \]

\[ 1 \lor \neg x \]

\[ \neg x \]
A one-generated infinite CIdInRL
A one-generated infinite CIdInRL
The fusion semilattice of a one-generated infinite CIdInRL
The fusion semilattice of a one-generated infinite CldInRL
The fusion semilattice of a one-generated infinite CldInRL
The fusion semilattice of a one-generated infinite CldInRL
The fusion semilattice of a one-generated infinite CldInRL
The fusion semilattice of a one-generated infinite CIdInRL
The fusion semilattice of a one-generated infinite CldInRL
The fusion semilattice of a one-generated infinite CldInRL
A **FL**\(^2\)-**algebra** is of the form \(A = (A, \land, \lor, \Diamond, \rightarrow, \leftarrow, t, f, \cdot, \backslash, /, 1, 0)\) s.t.

\[ A_t = (A, \land, \lor, \Diamond, \rightarrow, \leftarrow, t, f) \quad \text{and} \quad A_1 = (A, \land, \lor, \cdot, \backslash, /, 1, 0) \]

are pointed residuated lattices.

**Relation algebras** are examples of **classical** FL\(^2\)-algebras: \(A_t\) is a Boolean algebra with \(x \land y = x \Diamond y\).

A **bounded generalized bunched implication algebra** (bGBI-algebra) is a FL\(^2\)-algebra that satisfies \(x \land y = x \Diamond y\), \(t = \top\), \(f = \bot\) and \(0 = 1\).

A **bunched implication algebra**, or **BI-algebra**, is a commutative bGBI-algebra (i.e., \(xy = yx\)).
A **congruence filter** of a residuated lattice $A$ is a subset of the form $F = \uparrow([1]_\theta)$ where $\theta$ is a congruence.

Congruence filters satisfy the following **normality condition** for $a \in A$ (where quantifiers range over $F$):

$$\forall x \in F \exists x_1, x_2 \in F, \hspace{1em} x_1 a \leq ax \hspace{1em} \text{and} \hspace{1em} ax_2 \leq xa.$$  \hspace{1em} (N_a)

A filter $F$ satisfies $(N)$ if $(N_a)$ holds for all $a \in A$.

The set of **congruence filters** of $A$ is denoted by $\text{CF}(A)$.

**Theorem (Blount-Tsinakis 2003)**

For a residuated lattice $A$, a subset $F$ is a congruence-filter if and only if $F$ is a lattice filter and a submonoid of $A$ that satisfies $(N)$.

Moreover, $\text{Con}(A)$ is isomorphic to the lattice $\text{CF}(A)$ of congruence-filters via the bijection $\theta \mapsto \uparrow([1]_\theta)$ and $F \mapsto \{(x, y) : x/y, y/x \in F\}$.
Congruences of $\text{FL}^2$-algebras

For $\text{FL}^2$ the congruence 1-filters are determined by a stronger $t$-normality condition. For any $a \in A$

$$
\forall x \in F, \exists x_1, x_2, x_3, x_4 \in F, \ \
ax_1 \leq a \diamond xt, \ \ x_2 a \leq xt \diamond a, \ \ a \diamond x_3 t \leq xa, \ \ x_4 t \diamond a \leq ax
$$

$$(N_a^t)$$

A filter $F$ satisfies $(N_t^t)$ if $(N_a^t)$ holds for all $a \in A$.

**Theorem**

*For an $\text{FL}^2$-algebra $A$, a subset $F$ is the 1-filter of some congruence $\theta$ of $A$ if and only if $F$ is a lattice filter and $\cdot, 1$-submonoid of $A$ that satisfies $(N_t^t)$*

An analogous result holds for congruence $t$-filters $\uparrow([t]_\theta$ of $\text{FL}^2$-algebras.*
Congruences of GBI-algebras

The previous result specializes to generalized bunched implication algebras:

**Corollary**

The 1-filters of a GBI-algebra $A$ are the filter submonoids that are closed under the terms

$$u_a(x) = a \backslash (a \land x \top), \quad v_a(x) = (a \to xa) / \top \quad \text{and} \quad \rho_a(x) = ax / a,$$

A previously known characterization of the congruence classes of GBI-algebras used more complicated terms with two parameters.

Similar 1-parameter terms exist for congruence $\top$-filters of GBI-algebra.

**Theorem**

For an involutive GBI-algebra, a lattice filter $F$ is a $\top$-filter if and only if for all $x \in F$ it follows that $\neg \neg x$, $\neg \neg x$, $\sim (\top (\neg x) \top) \in F$. 
Weakening relation algebras

For a poset $P = (P, \leq)$, let $Wk(P) = \{ R \subseteq P^2 : \leq; R; \leq \subseteq R \}$. Relations in $Wk(P)$ are called **weakening closed relations** since

$$x \leq u \ R \ v \leq y \implies x \ R \ y$$

$\sim R := (R^c)^\frown = \{(y, x) \mid (x, y) \notin R\}$, the **complement-converse** of $R$.

Weakening relations are closed under **complement-converse**, **union**, **intersection**, **Heyting implication** $\rightarrow$ (=$\text{residual of intersection}$), **relation composition** $;$ and **residuals** $\setminus, / \text{ of composition}$.

$1 := \leq$ is a weakening relation and is the identity of composition.

The **full weakening relation algebra** on a poset $P$ is

$$Wk(P) = (Wk(P), \cap, \cup, \rightarrow, P^2, \emptyset, ;, \sim, 1, 0), \text{ where } 0 = \sim 1.$$ 

Representable weakening relation algebras $= V\{Wk(P) \mid P \text{ is a poset}\}$. 
Double division conuclei

An **interior operator** $\delta$ on a poset is an order-preserving map such that $\delta(\delta(x)) = \delta(x) \leq x$.

An interior operator $\delta$ is a **conucleus** if $\delta(x)\delta(y) \leq \delta(xy)$.

The conucleus **image** $\delta(A)$ of a residuated lattice is a residuated lattice $(\delta(A), \land, \lor, \cdot, \backslash_\delta, /_\delta)$ without 1, where $x *_\delta y = \delta(x * y)$ for $* \in \{\land, \backslash, /\}$.

Let $p \in A$ be a **positive idempotent**, i.e., $p = p^2 \geq 1$.

Then $\delta_p(x) = p\backslash x/p$ is a conucleus called the **double division conucleus**.

**Lemma**

$\delta_p(A) = \{pxp \mid x \in A\}$, and $p$ is the identity element.
Double division conuclei of relation algebras

In a full relation algebra, a positive idempotent $p$ is a **preorder** $P = (P, \sqsubseteq)$ (i.e., $p = \sqsubseteq$ is reflexive and transitive).

If $p \wedge p^\sim = 1$ then $P$ is a poset and $Wk(P) = \delta_p(\text{Rel}(P))$.

Hence the variety RWkRA of representable weakening relation algebras contains all double division conucleus images of members of RRA.

For a class $\mathcal{K}$ of algebras let $d\mathcal{K} = \{\delta_p(A) : A \in \mathcal{K}, 1 \leq p^2 = p \in A\}$.

**Theorem**

*If $\mathcal{V}$ is a variety of bounded GBI-algebras with $\top \setminus x/\top$ as unary discriminator on the subdirectly irreducible members then $S(d\mathcal{V})$ is a discriminator variety with the same unary discriminator term.*

Applying this result to the variety RA produces the discriminator variety $S(dRA)$ that contains both RA and RWkRA.
Some identities that hold in $S(\text{dRA})$

Recall that the variety $\text{RA}$ of relation algebras is an abstract counterpart of the variety $\text{RRA}$ of representable relation algebras.

The variety $S(\text{dRA})$ generated by double-division conucleus images of relation algebras is the abstract counterpart of $\text{RWkRA}$.

**Open problem:** Find a (finite?) axiomatization of $S(\text{dRA})$.

In a GBI-algebra let the **domain** $d(x) = x \top \land 1$ and **range** $r(x) = \top x \land 1$.

**Theorem**

The identities

\[
d(x)x = x, \quad xr(x) = x, \quad T x T x T = T x T \quad \text{and} \quad \sim \neg(xy) \leq (\sim \neg y)(\sim \neg x)
\]

hold in $S(\text{dRA})$. 


Thank you!