Primitive Lattice Varieties

Peter Jipsen and J. B. Nation

Chapman University and University of Hawai‘i

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Definitions and examples

A class $\mathcal{V}$ of algebras of the same type, e.g. lattices $(\mathbf{L}, \wedge, \vee)$, is a **variety** if $\mathcal{V} = \{\mathbf{L} : \mathbf{L} \models \mathcal{E}\}$ for some set of equations $\mathcal{E} = \{s_i = t_i : i \in I\}$

A class $\mathcal{K}$ of algebras of the same type is a **quasivariety** if $\mathcal{K} = \{\mathbf{L} : \mathbf{L} \models \mathcal{Q}\}$ for some set $\mathcal{Q}$ of quasiequations (i.e. formulas $\varepsilon_0 \& \cdots \& \varepsilon_{n-1} \Rightarrow \varepsilon_n$ where the ε_j are equations).

E.g. **commutative monoids** are a variety defined by $\mathcal{E} = \{(x \cdot y) \cdot z = y \cdot (x \cdot z), x \cdot 1 = x\}$

Cancellative commutative monoids are a (proper) (sub)quasivariety defined by $\mathcal{Q} = \mathcal{E} \cup \{x \cdot y = x \cdot z \Rightarrow y = z\}$

A variety is **primitive** if every subquasivariety is defined by equations, i.e., a subvariety.

Background

Again: a variety is **primitive** if every subquasivariety is defined by equations, i.e., a subvariety.

In algebraic logic, this property is called **hereditary structural completeness**

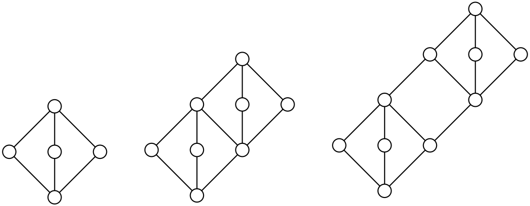
Example: The variety $\mathcal{D}$ of distributive lattices is primitive

Proof (hint): $\mathcal{D} = \mathbb{SP}$(any nontrivial distributive lattice)

The variety of commutative monoids is **not** primitive

Which other (lattice) varieties are primitive?

Two simple modular lattices and a subdirectly reducible one



$\mathbf{M}_3$ $\mathbf{M}_{3,3}$ $\mathbf{M}^{+}_{3,3}$

Which lattices generate primitive lattice varieties?

Lattices of subquasivarieties can be complicated

Let $\mathrm{L_q}(\mathcal{K})$ be the lattice of subquasivarieties of $\mathcal{K}$.

A quasivariety $\mathcal{K}$ is ***Q*-universal** if, for every quasivariety $\mathcal{Q}$ of finite type, $\mathrm{L_q}(\mathcal{Q})$ is a homomorphic image of a sublattice of $\mathrm{L_q}(\mathcal{K})$. If $\mathcal{K}$ is a *Q*-universal quasivariety, then

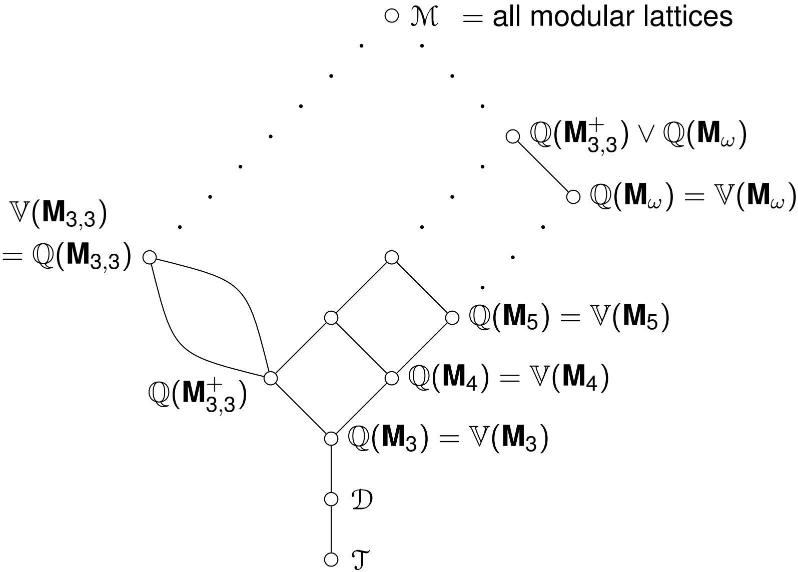
- $|\,\mathrm{L_q}(\mathcal{K})| = 2^{\aleph_0}$,
- the free lattice $\mathrm{FL}(\omega)$ is a sublattice of $\mathrm{L_q}(\mathcal{K})$, whence in particular $\mathrm{L_q}(\mathcal{K})$ satisfies no lattice identity.

It turns out that *Q*-universal quasivarieties are ubiquitous.

Theorem (Grätzer and Lakser 1979, Adams and Dziobiak 1994)

$\mathbb{V}(\mathbf{M}_{3,3})$ *is Q-universal (hence* ***not*** *primitive)*

On the other hand, $\mathbb{V}(\mathbf{M}_3)$ **is** primitive


$\mathcal{M}$ = all modular lattices
$\mathbb{Q}(\mathbf{M}_{3,3}^+) \vee \mathbb{Q}(\mathbf{M}_\omega)$
$\mathbb{Q}(\mathbf{M}_\omega) = \mathbb{V}(\mathbf{M}_\omega)$
$\mathbb{V}(\mathbf{M}_{3,3})$
$= \mathbb{Q}(\mathbf{M}_{3,3})$
$\mathbb{Q}(\mathbf{M}_5) = \mathbb{V}(\mathbf{M}_5)$
$\mathbb{Q}(\mathbf{M}_4) = \mathbb{V}(\mathbf{M}_4)$
$\mathbb{Q}(\mathbf{M}_{3,3}^+)$
$\mathbb{Q}(\mathbf{M}_3) = \mathbb{V}(\mathbf{M}_3)$
$\mathcal{D}$
$\mathcal{T}$

Primitive (quasi)varieties

An algebra A is **weakly projective** in a class $\mathcal{K}$ if whenever A is a homomorphic image of $B \in \mathcal{K}$, then A embeds in B.

Theorem (Slavík 1975, Gorbunov 1977)

*A locally finite quasivariety $\mathcal{Q}$ of finite type is primitive **iff** every finite $\mathcal{Q}$-subdirectly irreducible $A \in \mathcal{Q}$ is weakly projective in $\mathcal{Q}_{\mathrm{fin}}$.*

If $\mathcal{Q}$ is primitive, $\mathrm{L_q}(\mathcal{Q})$ is distributive.

An algebra A is **projective** in a (quasi)variety $\mathcal{K}$ if whenever $h : B \twoheadrightarrow A$ for $B \in \mathcal{K}$, then $\exists g : A \to B$ such that $h \circ g = id_A$.

Lattice tools

Jónsson's Lemma: If $\mathbf{L}$ is a finite lattice, then the subdirectly irreducible lattices in $\mathbb{V}(\mathbf{L})$ are contained in $\mathbb{HS}(\mathbf{L})$.

Whitman's condition (W):
$x \wedge y \leq z \vee w \ \Rightarrow\ x \leq z \vee w$ or $y \leq z \vee w$ or $x \wedge y \leq z$ or $x \wedge y \leq w$

Theorem (Davey and Sands 1977)

*Every finite lattice satisfying (W) is **projective** in the class of finite lattices.*

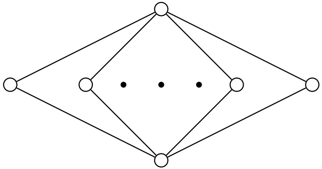
So now we can simplify the Slavik-Gorbunov characterization of primitive varieties for lattices.

Primitive lattice varieties

Call a finite lattice **L** **inherently Whitman** (IW) if every subdirectly irreducible lattice in $\mathbb{HS}(\mathbf{L})$ satisfies (W).

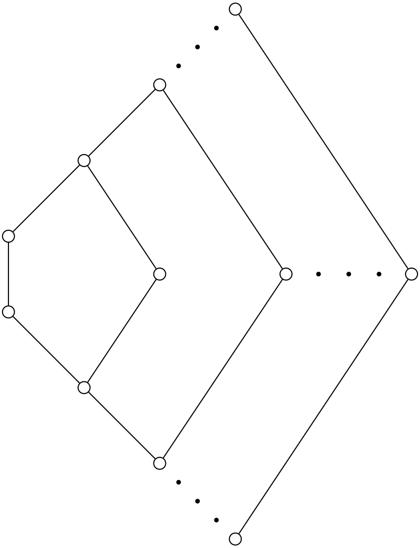
Corollary

If a finite lattice **L** *is* (IW) *then* $\mathbb{V}(\mathbf{L})$ *is a primitive variety.*



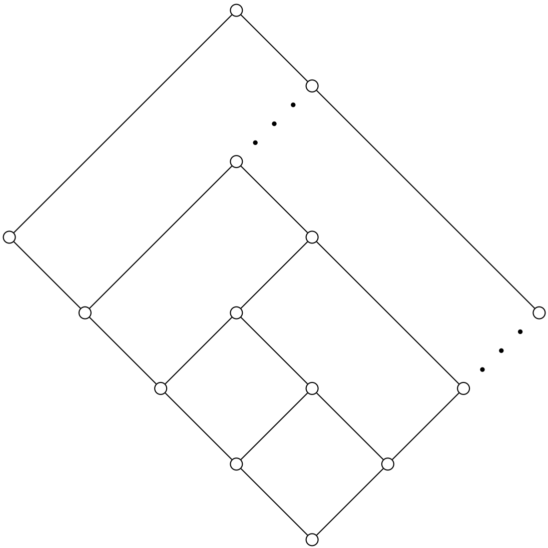
M_k

Lots more primitive lattice varieties



N_k

Lots more primitive lattice varieties



J_k

Comments on primitive lattice varieties

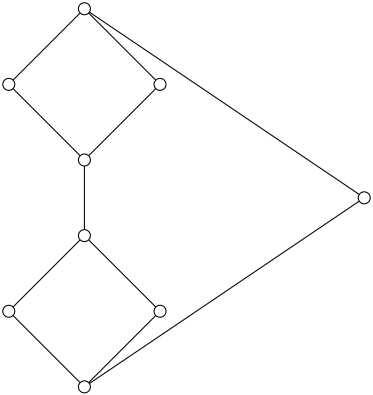
The previous slides show 3 infinite sequences of primitive lattice varieties. Many more can be obtained from certain glueing results.

Theorem

If $\mathbf{L}$ *is s.i. and satisfies* (W) *and* $|L| \leq 8$*, then* $\mathbf{L}$ *is inherently Whitman.*

Conjecture: If $\mathbf{L}$ is a finite subdirectly irreducible lattice satisfying (W), then $\mathbb{V}(\mathbf{L})$ is primitive.

WRONG! A 9-element counterexample



J

Subdirectly irreducible, satisfies (W), but $\mathbf{J}/\mu$ is s.i., does **not** satisfy (W)

A more detailed result

Theorem

*A locally finite lattice variety $\mathcal{V}$ is **not** primitive **iff** there exist finite $\mathbf{L}, \mathbf{L}' \in \mathcal{V}$ such that*

(1) $\mathbf{L}$ *is s.i.,*

(2) $\mathbf{L} \not\leq \mathbf{L}'$, *i.e.,* $\mathbf{L}'$ *does not contain* $\mathbf{L}$ *as a sublattice,*

(3) *there is a surjective homomorphism* $f : \mathbf{L}' \twoheadrightarrow \mathbf{L}$.

If $\mathcal{V}$ is not primitive, we may also assume that

(4) *there is a* (W)*-failure* $s \leq t$ *in* $\mathbf{L}$,

Results for proving a variety is not primitive

Corollary

Suppose a finite lattice $\mathbf{K}$ *satisfies* (W) *but isn't* IW, *i.e., there is a s.i.* $\mathbf{L}$ *in* $\mathbb{HS}(\mathbf{K})$ *that fails* (W). *Then* $\mathbb{V}(\mathbf{K})$ *is* ***not*** *primitive.*

Example: $\mathbb{V}(\mathbf{J})$ (the 9-elt IW counterexample) is **not** primitive

$\mathbf{L}[I]$ denotes Alan Day's doubling of interval I.

Corollary

If $\mathbf{L}$ *is s.i.,* I *is a* (W)*-failure interval in* $\mathbf{L}$*, and* $\mathbf{L}[I] \in \mathbb{V}(\mathbf{L})$*, then* $\mathbb{V}(\mathbf{L})$ *is* ***not*** *primitive.*

Example: $\mathbb{V}(\mathbf{M}_{3,3})$ is **not** primitive, since $\mathbf{M}_{3,3}^{+} = \mathbf{M}_{3,3}[I]$

Projective vs weakly projective

Problem: If a lattice is weakly projective in a locally finite variety, is it necessarily projective in that variety?

Theorem (Keith Kearnes, priv. comm. 2022)

Let $\mathbf{K}$ *be a finite s.i. algebra in a congruence distributive variety. If* $\mathbf{K}$ *is weakly projective in* $\mathbb{V}(\mathbf{K})$*, then* $\mathbf{K}$ *is projective in* $\mathbb{V}(\mathbf{K})$*.*

Characterization for finite (W)-lattices

The following result combines a previous theorem and corollary:

Theorem

Let $\mathbf{L}$ *be a finite lattice that satisfies* (W)*.*
Then $\mathbb{V}(\mathbf{L})$ *is primitive* ***iff*** $\mathbf{L}$ *is* IW*.*

A **variety is IW** if it is locally finite and every finite s.i. in it is IW.

By Jonsson's Lemma, an s.i. $\mathbf{K} \in \mathcal{V} \vee \mathcal{W}$ iff $\mathbf{K} \in \mathcal{V} \cup \mathcal{W}$, hence IW varieties form a lattice ideal.

Theorem

Finitely generated, inherently Whitman varieties form an ideal in the lattice Λ *of lattice varieties.*

Sequences of inherently Whitman lattices

Theorem

Assume that $\mathbf{S}_k$ $(k \in \omega)$ *form a chain of* $\mathbb{IW}$ *lattices:*

$$\mathbf{S}_0 \leq \mathbf{S}_1 \leq \mathbf{S}_2 \leq \dots$$

Let $\mathbf{S}_\omega = \bigcup_{k\in\omega} \mathbf{S}_k$.

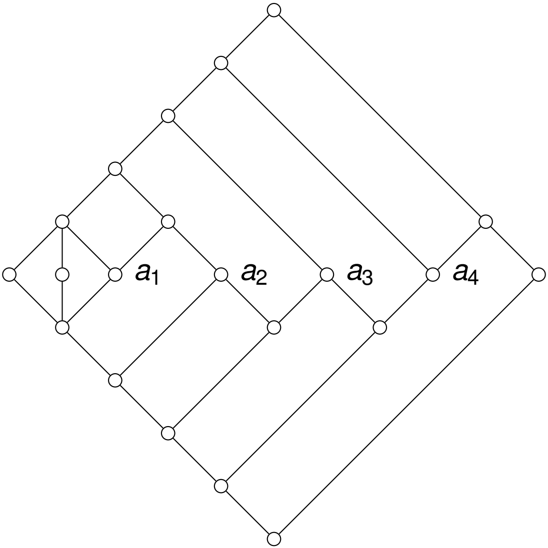
If $\mathbb{V}(\mathbf{S}_\omega)$ *is locally finite, then it is a primitive lattice variety.*

Using appropriate sequences of lattices, we get:

Corollary

There are $2^{\aleph_0}$ *primitive lattice varieties.*

One of the S_k



Some primitive lattice varieties that fail (W)

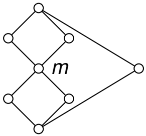
Theorem

Let $\mathbf{L}$ *be a finite s.i. lattice with a doubly reducible element* m *that has exactly* 2 *upper covers* a, b *and* $m = a \wedge b$ *is the unique irredundant proper meet decomposition of* m.
If, in addition,

1. $\mathbf{L}[m] \notin \mathbb{V}(\mathbf{L})$, *and*
2. *every s.i. lattice in* $\mathbb{HS}(\mathbf{L}) \setminus \{\mathbf{L}\}$ *is weakly projective in* $\mathbb{V}(\mathbf{L})$,

then $\mathbb{V}(\mathbf{L})$ *is a primitive lattice variety.*

For example $\mathbb{V}(\mathbf{J}/\mu)$ is primitive



$\mathbf{J}/\mu$

Checking if $\mathcal{V}$ is primitive

An (incomplete) test to answer the question: **Is $\mathcal{V}$ primitive?**

(A) If $\mathcal{V}$ is IW, then YES.

(B) If $\mathcal{V}$ contains a finite lattice **L** that satisfies (W) but **L** is not IW, i.e., there exists a s.i. $\mathbf{K} \in \mathbb{HS}(\mathbf{L})$ that fails (W), then NO.

(C) If there is a s.i. **K** in $\mathcal{V}$ which has a (W)-failure interval I such that $\mathbf{K}[I] \in \mathcal{V}$, then NO.

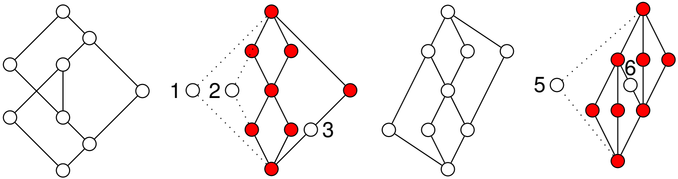
(D) If for every s.i. **K** in $\mathcal{V}$, either **K** satisfies (W) or there is a single (W)-failure interval I of **K**, such that the property $\varpi(\mathbf{K}, \mathbf{K}[I])$ holds and $\mathbf{K}[I] \notin \mathcal{V}$, then YES.

(E) If for every s.i. **K** in $\mathcal{V}$, either **K** satisfies (W) or there is a lattice $\mathbf{M} \notin \mathcal{V}$ such that the property $\varpi(\mathbf{K}, \mathbf{M})$ holds, then YES.

$\varpi(\mathbf{K}, \mathbf{M})$: for any finite **L**, $f : \mathbf{L} \twoheadrightarrow K$ implies $\mathbf{K} \leq \mathbf{L}$ or $\mathbf{M} \leq \mathbf{L}$

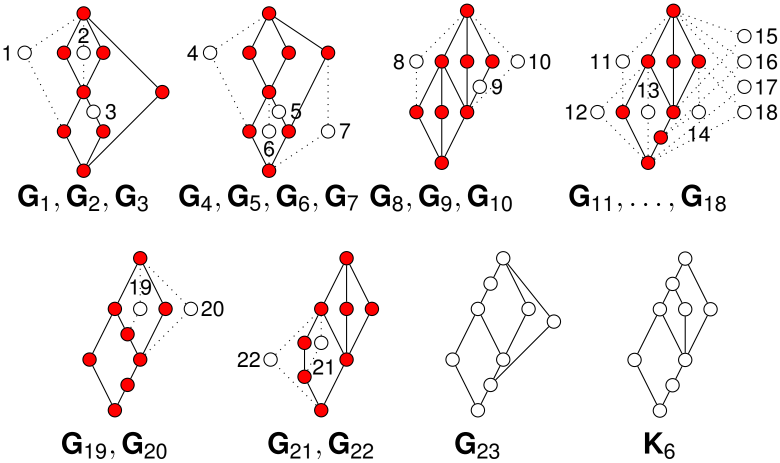
The number of s.i. lattices of cardinality ≤ 12 that fail (W):

Cardinality $n =$	5	6	7	8	9	10	11	12
Number of s.i.'s	2	4	16	69	360	2103	13867	100853
s.i.'s that fail (W)	0	0	0	4	55	629	6360	61634



$\mathbf{P}_5$ $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ $\mathbf{H}_4$ $\mathbf{H}_5, \mathbf{H}_6$

All 7 self-dual s.i. lattices with 9 elements that fail (W). For $\mathbf{H}_i$ $(i = 1, \ldots, 6)$ add element i, possibly connected by dotted lines.



All 24 non-self-dual s.i. lattices with 9 elements that fail (W).
For $\mathbf{G}_i$ $(i = 1, \ldots, 23)$ add element i, possibly connected by dotted lines. The dual lattices are $\mathbf{G}_1^d, \ldots, \mathbf{G}_{23}^d, \mathbf{K}_6^d$.

Primitive varieties generated by 9-element lattices

There are 360 s.i. lattices with nine elements.

305 satisfy (W) and all but one are IW, hence generate a primitive variety.

The 9-elt counterexample is the exception, and does NOT generate a primitive variety.

There are 55 nine-element s.i. lattices that fail (W).

Some of those 55 that fail (W) contain $\mathbf{M}_{3,3}$, and those automatically generate a non-primitive variety.

However, 16 s.i. lattices with 9 elements DO generate a primitive variety: the self-dual lattices $\mathbf{P}_5$, $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, and the non-self-dual $\mathbf{G}_j$ $(1 \leq j \leq 5)$ and $\mathbf{G}_{23}$.

Conclusion and questions

Primitive variety means every subquasivariety is a subvariety.

Every locally finite, inherently Whitman lattice variety is primitive

There are uncountably many of that type.

There are also primitive lattice varieties $\mathbb{V}(\mathbf{L})$ where $\mathbf{L}$ fails Whitman's condition (W).

Some interesting problems remain.

1. An earlier (more detailed) result showed that it is decidable whether a finite lattice generates a primitive variety. Is there a reasonably efficient algorithm? Is there a reasonable algorithm at least to decide whether $\mathbb{V}(\mathbf{L})$ is primitive when $\mathbf{L}$ satisfies (W)?
2. If $\mathbb{V}(\mathbf{L})$ fails to be primitive, is it Q-universal as a quasivariety?

References

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Thanks!