Structural results about involutive residuated lattices and involutive po-monoids

José Gil-Férez, Peter Jipsen* and Sid Lodhia

Chapman University

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Involutive residuated lattices = involutive semirings

An **involutive residuated lattice**, or **involutive semiring**, is an algebra $\mathbf{A} = (A, \vee, \cdot, 1, \sim, -)$ such that

- $(A, \vee)$ is a join-semilattice with order $x \leqslant y \iff x \vee y = y$,
- $(A, \cdot, 1)$ is a monoid, and
- $x \leqslant y \iff x \cdot {\sim}y \leqslant -1 \iff -y \cdot x \leqslant -1$ $(ineg)$

The unary operations $\sim$ and $-$ are called **involutive negations**, and $\mathbf{A}$ is a lattice with $x \wedge y = {\sim}(-x \vee -y)$.

$-1 = {\sim}1$ is denoted by 0, and we often write xy instead of $x \cdot y$.

The class **InRL** of all involutive residuated lattices is a variety that includes all ℓ-groups (where ${\sim}x = -x = x^{-1}$ so $0 = 1^{-1} = 1$).

Integral InRLs have 1 as top element, i.e., $x \vee 1 = 1$ so $x \wedge 0 = 0$.

Examples of integral involutive residuated lattices

Given an ℓ-group $\mathbf{L} = (L, \vee, \cdot^{\mathbf{L}}, ^{-1}, 1)$ one can construct **integral InRLs** from any interval $[z, 1] \subseteq L$ by defining

$$0 = z, \;\; xy = (x \cdot^{\mathbf{L}} y) \vee 0, \;\; {\sim}x = x^{-1}0 \;\text{ and }\; -x = 0x^{-1}.$$

This is known as the **Mundici functor** and is a composition of the negative-cone nucleus and the z-conucleus.

From the ℓ-group $(\mathbb{Z}, \vee, +, -, 0)$ it produces finite MV-chains $Ł_n$.

The Chang algebra $\mathbf{C}$ is obtained from $\mathbb{Z} \times_{\text{lex}} \mathbb{Z}$ using $z = (-1, 0)$.

One can construct many other integral InRLs from rotations and products, but in general there is no full description of (even the finite) integral InRLs.

Constructing nonintegral InRLs from integral components

The aim of this talk is to explain very recent results about how to construct a large **subvariety of InRLs** from collections of integral InRLs as building blocks.

This produces a **full structural description** of these algebras in terms of their **integral components**.

Previously a **restricted version** of this structural description was obtained by **[J., Tuyt and Valota 2020]** for **finite commutative idempotent InRLs**.

In this case the integral components are **finite Boolean algebras**, i.e., finite powers of **2**.

In the present setting the components are **arbitrary** integral InRLs.

Locally integral involutive residuated lattices

The algebras that are constructed from integral components satisfy **three** simple identities:

$$x \cdot {\sim}x = -x \cdot x \quad \text{and} \quad x \wedge 0 \leqslant x^2 \leqslant x$$

InRLs that satisfy these identities are called **locally integral**, and the variety of all of them is denoted **LIInRL**.

Integral InRLs are **locally integral**: $x \cdot {\sim}x = 0 = -x \cdot x$ and $x \wedge 0 = 0 \leqslant x^2 \leqslant x1 = x$.

The main result is that **all** locally integral InRLs can be constructed from a **disjoint union of integral InRLs** and a certain system of $\vee, \cdot, 1$-homomorphisms.

Involutive po-monoids

The result applies more generally to partially ordered structures:

An **involutive partially ordered monoid**, or **ipo-monoid** for short, is a structure $(A, \leqslant, \cdot, 1, \sim, -)$ such that

- $(A, \leqslant)$ is a poset,
- $(A, \cdot, 1)$ is a monoid, and
- $x \leqslant y \iff x \cdot {\sim}y \leqslant -1 \iff -y \cdot x \leqslant -1 \;\; (ineg)$

As before, $-1 = {\sim}1$ is denoted by 0.

The class of ipo-monoids includes **all groups** (if $\leqslant$ is $=$) and

all partially ordered groups (where ${\sim}x = -x = x^{-1}$).

Also, MV-algebras are ipo-monoids ($\vee, \wedge$ are definable).

Properties of involutive po-monoids

Lemma

Every ipo-monoid satisfies the following properties.

double negation:	$\sim\!-x = x = -\!\sim\!x$
rotation (rota):	$x \cdot y \leqslant z \iff y \cdot \sim z \leqslant \sim x \iff -z \cdot x \leqslant -y$
antitonicity:	$x \leqslant y \iff \sim y \leqslant \sim x \iff -y \leqslant -x$
residuation (res):	$xy \leqslant z \iff x \leqslant -(y \cdot \sim z) \iff y \leqslant \sim(-z \cdot x)$
constants:	$0 = \sim 1, \quad \sim 0 = 1 \quad$ *and* $\quad -0 = 1.$

(res) provides the **residuals** $z/y = -(y \cdot \sim z)$ and $x \backslash z = \sim(-z \cdot x)$.

It follows from (res) that $\cdot$ is **order-preserving** in both arguments.

Idempotent elements in ipo-monoids

An element a of an ipo-monoid A is **idempotent** if $a^2 = a$.

Lemma

Every ipo-monoid satisfies the following properties:

1. $-({\sim}x \cdot {\sim}y) = {\sim}(-x \cdot -y)$,
2. ${\sim}x$ *is idempotent if and only if* $-x$ *is idempotent.*

Proof.

1. Exercise.
2. Assume that ${\sim}x$ is idempotent. Then,
   $-x \cdot -x = -{\sim}(-x \cdot -x) = --({\sim}x \cdot {\sim}x) = --{\sim}x = -x$.
   The other implication is analogous.

□

An important equivalence for ipo-monoids

Lemma

In every ipo-monoid A, the following conditions are equivalent:

1. $-x \cdot x = x \cdot {\sim}x$ *holds in* A,
2. ${\sim}(-x \cdot x) = -(x \cdot {\sim}x)$ *holds in* A.

Proof.

Suppose that the equation $-x \cdot x = x \cdot {\sim}x$ holds in A. In particular, we have that $-{\sim}x \cdot {\sim}x = {\sim}x \cdot {\sim}{\sim}x$, that is, $x \cdot {\sim}x = {\sim}x \cdot {\sim}{\sim}x$. Hence,

$$ {\sim}(-x \cdot x) = {\sim}(-x \cdot -{\sim}x) = -({\sim}x \cdot {\sim}{\sim}x) = -(x \cdot {\sim}x) $$

where the middle equality follows by (1) of the preceding Lemma. (The proof of the reverse implication is omitted.) □

Positive elements

Let $A^+ = \{x \in A \mid 1 \leqslant x\}$ be the **positive cone** of A.

Using residuals, $\sim(-x \cdot x) = -(x \cdot \sim x)$ says $x\backslash x = x/x$.

Since $1x = x$, we have $1 \leqslant x/x$, hence the terms above are all in the positive cone.

These terms are **important** enough to introduce **special notation**:

$$1_x = -(x \cdot \sim x) \quad \text{and} \quad 0_x = x \cdot \sim x.$$

If we assume $x \cdot \sim x = -x \cdot x$ then $1_x = \sim 0_x = -0_x$.

Locally integral ipo-monoids

An ipo-monoid is **integral** if it satisfies the inequality $x \leqslant 1$.

Since $\sim, -$ are dual isomorphisms, integrality implies $0 \leqslant x$.

An ipo-monoid is **locally integral** if it satisfies

1. $-x \cdot x = x \cdot {\sim}x$,
2. multiplication is square-decreasing, i.e., $x^2 \leqslant x$,
3. $\downarrow$0-idempotence, i.e., $x \leqslant 0 \implies x^2 = x$.

Proposition

Every integral ipo-monoid is **locally integral**.

Properties of locally integral ipo-monoids

Define $A_x = \{y \in A : 1_x = 1_y\} = \{y \in A : 0_x = 0_y\}$.

Lemma

Let $\mathbf{A}$ *be a locally integral ipo-monoid. For all* x *and* y *in* A*, the following properties hold:*

1. $0_{\sim x} = 0_{-x} = 0_x$ *and* $1_{\sim x} = 1_{-x} = 1_x$,
2. $x \in [0_x, 1_x]$ (*where* $[0_x, 1_x] = \{a \in A \colon 0_x \leqslant a \leqslant 1_x\}$),
3. $1_x \cdot y = y \iff 1_x \leqslant 1_y$,
4. $y \in [0_x, 1_x] \iff [0_y, 1_y] \subseteq [0_x, 1_x]$,
5. $y \in A_x \iff y \in [0_x, 1_x]$ *and* $1_x \cdot y = y$.

Canonical representatives for the A_x equivalence classes

$x \equiv y \iff 1_x = 1_y$ is an **equivalence relation**.

Hence the equivalence classes A_x **partition** A.

Lemma

Let $\mathbf{A}$ *be a locally integral ipo-monoid.*

1. *For all* p *and* a *in* A*, we have that* $p \in A^+ \iff p = 1_p$ *and* $a \in {\downarrow}0 \iff a = 0_a$*.*
2. *For every* x *in* A*,* 1_x *is the only positive element of* A_x *and* 0_x *is the only element of* A_x *below* 0*.*

It follows that the classes A_x are **indexed** by the elements of A^+.

The integral components of locally integral ipo-monoids

Theorem

Let $\mathbf{A}$ *be a locally integral ipo-monoid. For every* p *in* A^+*,*

1. A_p *is closed under* $\sim, -, \cdot$ *and under all existing nonempty joins and nonempty meets, and*
2. *the structure* $\mathbf{A}_p = (A_p, \leqslant, \cdot, 1_p, \sim, -)$ *is an integral ipo-monoid, where* $\leqslant, \cdot, 1_p, \sim, -$ *are restricted to* A_p*,*
3. *if* $\mathbf{A}$ *is a LIInRL then* $\mathbf{A}_p$ *is an integral InRL.*

The structures $\mathbf{A}_p$ are called the **integral components** of $\mathbf{A}$.

Positive elements are central

Proposition

All positive elements of a locally integral ipo-monoid are central.

Proof.

Suppose that p is positive and let x be an arbitrary element. The inequality $(p \cdot -(px)p) \cdot x = p \cdot (-(px) \cdot px) = p(px \cdot {\sim}(px)) = ppx \cdot {\sim}(px) = px \cdot {\sim}(px) = 0_{px} \leqslant 0$ implies that $p \cdot -(px) \leqslant p \cdot -(px)p \leqslant -x$. By (rota), $-(px)x \leqslant {\sim}p = -p$, and by (rota) again, $xp \leqslant px$.

Applying (rota) to $xp \leqslant xp$, we obtain that $-(xp)x \leqslant -p = {\sim}p$, and by (rota) again, $p \cdot -(xp) \leqslant -x$. Finally, since $xp \leqslant px$ is true for any x, in particular we have that $-(xp)p \leqslant p \cdot -(xp) \leqslant -x$, and by (rota) one last time, $px \leqslant xp$. □

Products between components and distributivity of $\downarrow 0$, A^+

Lemma

Given a locally integral ipo-monoid, two positive elements p and q, and two elements $x \in A_p$ and $y \in A_q$, their product xy is in A_{pq}.

Moreover, $0_p \cdot 0_q = 0_{pq}$ and $1_p \cdot 1_q = 1_{pq}$.

The idempotence of all $p, q \in A^+$ implies $pq = p \vee q$.

By duality, $y, z \leqslant 0$ implies $yz = y \wedge z$.

If $\downarrow 0$ is finite then joins exist, and since $\cdot$ distributes over $\vee$, $\downarrow 0$ is a distributive lattice, and by duality A^+ is a distributive lattice.

Płonka sums

The previous results indicate that the $\cdot$ operation of locally integral ipo-monoids are built from $\cdot$ in the integral components.

A family φ_{ij} of homomorphisms is **compatible** if for every $i \in I$, φ_{ii} is the identity on $\mathbf{A}_i$, and if $i \leqslant j \leqslant k$ then $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$.

Given a compatible family of homomorphisms between algebras $\{\varphi_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j : i \leqslant j\}$, indexed by the order of a lower-bounded join-semilattice $(I, \vee, \bot)$, its **Płonka sum** is the disjoint union $A = \biguplus_{i \in I} A_i$, where for every constant symbol c, $c^{\mathbf{A}} = c^{\mathbf{A}_\bot}$, and for every n-ary operation symbol σ, elements $a_1 \in A_{i_1}, \ldots,$ $a_n \in A_{i_n}$ and $j = i_1 \vee \cdots \vee i_n$ we have

$$\sigma^{\mathbf{A}}(a_1, \ldots, a_n) = \sigma^{\mathbf{A}_j}(\varphi_{i_1 j}(a_1), \ldots, \varphi_{i_n j}(a_n)).$$

Compatible maps between components

For positive $p \leqslant q$, **define** $\varphi_{pq} : A_p \to A_q$ by $\varphi_{pq}(x) = qx$.

Proposition

Let $\mathbf{A}$ be a locally integral ipo-monoid and $p \leqslant q$ two positive elements. The map $\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q$ is well defined and it is a monoidal homomorphism. Moreover, it respects arbitrary nonempty existing joins. In particular, φ_{pq} is monotone.

$\{\varphi_{pq} : p \leqslant q\}$ is a compatible family of monoidal homomorphisms indexed by the order of the join semilattice $(A^+, \cdot, 1)$.

$$\varphi_{pq}(xy) = qxy = qqxy = qxqy = \varphi_{pq}(x)\varphi_{pq}(y).$$

Structural characterization of locally integral ipo-monoids

Theorem

Let $\mathbf{A}$ *be a locally integral ipo-monoid and* $\{\varphi_{pq} : p \leqslant q\}$ *the family of monoidal homomorphism as before.*
Then the Płonka sum $\left(\biguplus A_p, \cdot^S, 1^S\right)$ *is the monoidal reduct of* $\mathbf{A}$.

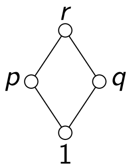
Define ${\sim^S} x = {\sim^{A_p}} x$ *and* $-^S x = -^{A_p} x$, *for every* $x \in A_p$ *with* p *positive, and*

$$x \leqslant^S y \iff x \cdot^S {\sim^S} y = 0_{pq}, \quad \text{for all } x \in A_p \text{ and } y \in A_q.$$

Then $\left(\biguplus A_p, \leqslant^S, \cdot^S, \sim^S, -^S\right)$ *is* $\mathbf{A}$.

If $\mathbf{A}$ *is in* **InRL** *then all* $\mathbf{A}_p$ *are in* **InRL**.
$\mathbf{A}$ *is commutative if and only if* $\mathbf{A}_p$ *is commutative for all* p *in* A^+.

A generic example with 4 components



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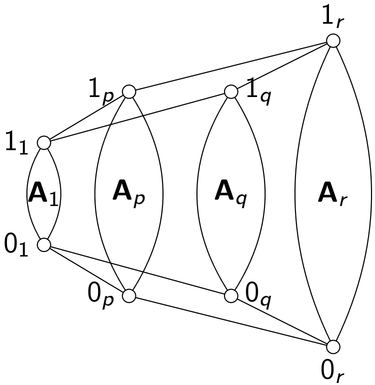
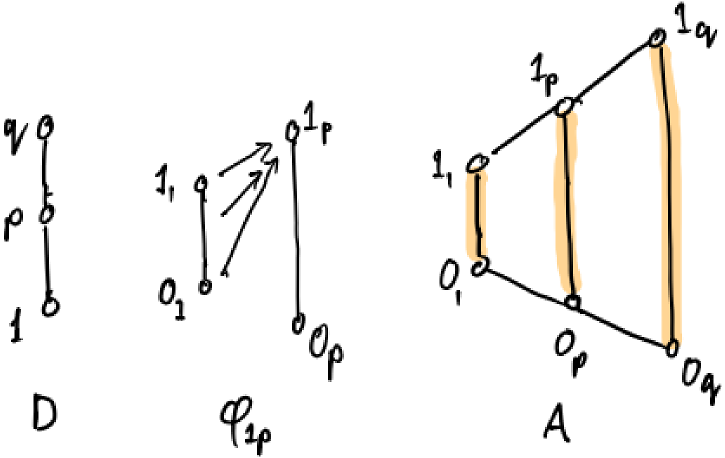
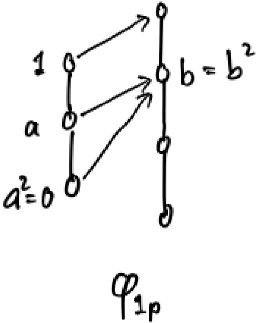
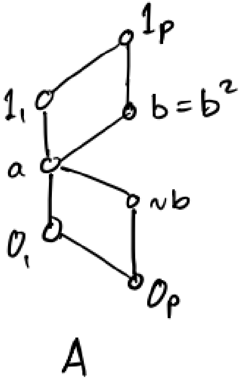


A Sugihara glueing of copies of the standard MV-chain



Glueing $Ł_3$ into a small IMTL-algebra





How many small ipo-monoids can be constructed this way?

Models of cardinality $n =$	1	2	3	4	5	6	7	8	9
Integral ipo-monoids	1	1	1	3	3	13	17	84	145
Locally integral ipo-monoids	1	1	2	5	9	28	57	194	448
Integral InRLs	1	1	1	3	3	12	17	78	140
Locally integral InRLs	1	1	2	5	9	26	54	171	404
[J., Tuyt, Valota 2020]									
Boolean algebras	1	1	0	1	0	0	0	1	0
Comm. Idempotent InRLs	1	1	1	2	2	4	4	9	10

So ipo-monoids can be **reconstructed** from their components.

But if we are given a compatible system φ_{pq} of $(D, \vee, 1)$-indexed monoidal homomorphisms between integral ipo-monoids $\mathbf{A}_p$, **what conditions have to hold** for the construction of a locally integral ipo-monoid G?

Glueing integral ipo-monoids

Let $(D, \vee, 1)$ be a lower-bounded join-semilattice such that for every $p \in D$, $\mathbf{A}_p = (A_p, \leqslant_p, \cdot_p, 1_p, \sim_p, -_p)$ is an integral ipo-monoid, and $\Phi = \{\varphi_{pq} : p \leqslant^D q\}$ is a compatible family of monoidal homomorphism $\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q$. Define the structure

$$\int_\Phi \mathbf{A}_p = \left(\biguplus_D A_p, \leqslant^G, \cdot^G, 1^G, \sim^G, -^G \right)$$

where $\left(\biguplus_D A_p, \cdot^G, 1^G \right)$ is the Płonka sum of the family Φ, and therefore a monoid, and for all $p, q \in D$, $a \in A_p$, and $b \in A_q$, $\sim^G a = \sim_p a$ and $-^G a = -_p a$, and

$$a \leqslant^G b \iff a \cdot^G \sim^G b = 0_{p \vee q}.$$

$\int_\Phi \mathbf{A}_p$ is the **glueing of** $\{A_p : p \in D\}$ **along the family** Φ.

Required conditions for glueing integral ipo-monoids

(bal)anced: for all $p, q \in D, a \in A_p, b \in A_q$,

$$a \cdot^G \sim^G b = 0_{p \vee q} \iff -^G b \cdot^G a = 0_{p \vee q}.$$

(zero): for all $p \leqslant^D q$, $\quad \varphi_{pq}(0_p) = 0_q \iff p = q$.

(tr): for all $a, b, c \in \biguplus A_p$, $\quad$ if $a \leqslant^G b$ and $b \leqslant^G c$, then $a \leqslant^G c$.

Main glueing result

Theorem

A structure $\mathbf{A}$ *is a locally integral ipo-monoid if and only if there is*

- *a lower-bounded join-semilattice* $\mathbf{D}$,
- *a family of integral ipo-monoids* $\{\mathbf{A}_p : p \in D\}$, *and*
- *a compatible family* $\Phi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant^D q\}$ *of monoidal homomorphisms satisfying (bal), (zero), (tr) so that* $\mathbf{A} = \int_\Phi \mathbf{A}_p$.

Can (tr) be replaced by a more "local" condition?

Find conditions that ensure $\mathbf{A}$ is lattice-ordered.

Are locally integral ipo-monoids or InRLs decidable?

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THANKS!