Bunched implication algebras, Heyting algebras with a residuated unary operator and their Kripke semantics

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Aim: find good structural descriptions of finite algebras

Good: every finite abelian group is ($\cong$ to) a product of cyclic groups $\mathbb{Z}_n$

Good: every finite Boolean algebra is ($\cong$ to) the set of all subsets of a set

Good: every finite MV-algebra is ($\cong$ to) a product of MV-chains $Ł_n$

The **fine spectrum** f_n of a class of algebras is the number of algebras of size n in the class (up to isomorphism).

Good structural descriptions give a formula for the fine spectrum.

finite abelian groups: $f_n = 1, 1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 2, \ldots$ = number of factorizations of n into prime powers.

finite MV-algebras: $f_n = 1, 1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 4, \ldots$ = number of ways of factoring n into a product with nontrivial factors.

finite Boolean algebras: $f_n = 1, 1, 0, 1, 0, 0, 0, 1, 0, \cdots = \begin{cases} 1 & \text{if } n = 2^k \\ 0 & \text{otherwise} \end{cases}$

finite fields: $f_n = 0, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, \cdots = \{1$ iff n is a prime power.

Classes without good structural descriptions

Finite semigroups $f_{10} = 12\,418\,001\,077\,381\,302\,684$, $f_{11} =?$

Finite lattices (or semilattices) $f_{20} = 23\,003\,059\,864\,006$ $(= f_{19})$, $f_{21} =?$

Finite groups $f_{1024} = 49\,487\,365\,422$, $f_{2048} =?$
the **classification of finite simple groups** is a milestone

Finite rings (with / without identity or commutativity) $f_{16} = 390$, $f_{32} =?$

Finite semirings with $0, 1$ $f_6 = 3246$, $f_7 =?$

Finite idempotent semirings "=" residuated lattices $f_8 = 295\,292$, $f_9 =?$

Heyting algebras and bunched implication algebras

Definition

A **Heyting algebra** $(A, \wedge, \vee, \bot, \top, \rightarrow)$ is a bounded lattice $(A, \wedge, \vee, \bot, \top)$ such that $\rightarrow$ is the residual of $\wedge$, i. e.,

$$x \wedge y \leq z \quad \iff \quad y \leq x \rightarrow z.$$

The residual $\rightarrow$ ensures that the lattice is **distributive**.

Definition

A **bunched implication algebra** (BI-algebra) $(A, \wedge, \vee, \bot, \top, \rightarrow, *, 1, -\!*)$ is a Heyting algebra $(A, \wedge, \vee, \bot, \top, \rightarrow)$ such that $(A, *, 1)$ is a commutative monoid and $-\!*$ is the residual of $*$, i. e.,

$$x * y \leq z \quad \iff \quad y \leq x -\!* z.$$

The class of Heyting algebras and BI-algebras can both be defined by equations, so they are **varieties**.

BI-algebras and BI-logic

Bunched implication algebras are the algebraic semantics of **BI-logic**

BI-logic is the propositional part of **separation logic**, which is a Hoare logic for reasoning about data structures, memory allocation and concurrent programs.

The structure of BI-algebras is not well understood.

Defining $\neg x = x \to \bot$ and adding $\neg\neg x = x$ to BI-algebras gives the variety of Boolean BI-algebras, which contains the variety CRA of commutative relation algebras.

Finite BI-algebras "=" finite commutative distributive residuated lattices.

Aim: find easy-to-describe subvarieties of BI-algebras

A BI-algebra is **idempotent** if $x{*}x = x$.

Recall that a **preorder** P is a binary relation that is reflexive and transitive

Theorem (Alpay, J. 2020)

Every finite idempotent **Boolean** *BI-algebra is determined by a preorder P on the set of atoms such that*

P is a preorder forest: xPy and xPz implies yPz or zPy, and

P has singleton roots: xPy and yPx and $\forall z(xPz \implies zPx)$ implies $x = y$

Preorder forests with singleton roots are counted by an Euler transform:

n	1	2	3	4	5	6	7	8	9	10	11
f_n	1	2	5	14	41	127	402	1306	4314	14465	49054
idem. BBI f_n	1	1	0	2	0	0	0	5	0	0	0

Now: Generalize to distributive lattice-ordered magmas

Definition

A **distributive lattice-ordered magma** ($d\ell$-**magma** for short) $(A, \wedge, \vee, \bot, \top, \cdot)$ is a bounded distributive lattice with a binary operation $\cdot$ such that for all $x, y, z \in A$

$$x\cdot(y \vee z) = x\cdot y \vee x\cdot z \qquad x\cdot\bot = \bot$$
$$(x \vee y)\cdot z = x\cdot z \vee y\cdot z \qquad \bot\cdot x = \bot$$

A $d\ell$-**monoid** is a $d\ell$-magma with $1 \in A$ such that $(A, \cdot, 1)$ is a monoid.

Every BI-algebra has a commutative $d\ell$-monoid as a reduct.

Every finite commutative $d\ell$-monoid expands uniquely to a BI-algebra.

Unary-determined $d\ell$-magmas

Definition

A $d\ell$-magma is **unary-determined** if $x{\cdot}y = (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$.

A **Boolean magma** is a $d\ell$-magma that has a complement operation $\neg$ s.t.

$$x \wedge \neg x = \bot \quad \text{and} \quad x \vee \neg x = \top.$$

Theorem

Every idempotent Boolean magma is unary-determined.

Proof.

Idempotence implies $x \wedge y \leq x{\cdot}y \leq x \vee y$ since $(x \wedge y)^2 \leq x{\cdot}y \leq (x \vee y)^2$ holds for order-preserving binary operations. Now

$$\begin{aligned} x{\cdot}\top \wedge y &= x{\cdot}(y \vee \neg y) \wedge y = (x{\cdot}y \wedge y) \vee (x{\cdot}(\neg y)) \wedge y) \\ &\leq x{\cdot}y \vee ((x \vee \neg y) \wedge y) = x{\cdot}y \vee (x \wedge y) \vee (\neg y \wedge y) = x{\cdot}y \end{aligned}$$

Idempotent Boolean BI-algebras are unary-determined

Proof (continued).

Similarly $x \wedge \top y \leq x{\cdot}y$, hence $x{\cdot}y \geq (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$.

The opposite inequality $x{\cdot}y \leq (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$ is equivalent to

$$x{\cdot}y \wedge \neg(x{\cdot}\top \wedge y) \leq x \wedge \top{\cdot}y$$

$$\iff (x{\cdot}y \wedge \neg(x{\cdot}\top)) \vee (x{\cdot}y \wedge \neg y) \leq x \wedge \top{\cdot}y$$

$$\iff (x{\cdot}y \wedge \neg y) \leq x \wedge \top{\cdot}y \quad \text{since } x{\cdot}y \leq x{\cdot}\top$$

By idempotence, $x{\cdot}y \wedge \neg y \leq (x \vee y) \wedge \neg y = (x \wedge \neg y) \vee (y \wedge \neg y) \leq x$ and $x{\cdot}y \wedge \neg y \leq x{\cdot}y \leq \top{\cdot}y$. □

Corollary

All idempotent Boolean BI-algebras are unary-determined.

Term-equivalence for unary-determined $d\ell$-magmas

Definition

A $d\ell pq$-**algebra** $(A, \wedge, \vee, \bot, \top, p, q)$ is a bounded distributive lattice with two unary operations p, q that satisfy

$$p\bot = \bot \qquad p(x \vee y) = px \vee py \qquad x \wedge p\top \leq qx$$
$$q\bot = \bot \qquad q(x \vee y) = qx \vee qy \qquad x \wedge q\top \leq px$$

Unary-determined $d\ell$-magmas are term-equivalent to $d\ell pq$-algebras:

Theorem

1. *Let* $\mathbf{A}$ *be a* $d\ell pq$*-algebra and define* $x{\cdot}y = (px \wedge y) \vee (x \wedge qy)$*. Then* $(A, \wedge, \vee, \bot, \top, \cdot)$ *is a* $d\ell$*-magma that is unary-determined and* p, q *are definable as* $px = x{\cdot}\top$ *and* $qx = \top{\cdot}x$*.*
2. *Let* $\mathbf{A}$ *be a unary-determined* $d\ell$*-magma and define* $px{=}x{\cdot}\top$*,* $qx{=}\top{\cdot}x$*. Then* $(A, \wedge, \vee, \bot, \top, p, q)$ *is a* $d\ell pq$*-algebra and* $\cdot$ *is definable as* $x{\cdot}y = (px \wedge y) \vee (x \wedge qy)$*.*

Associativity, Commutativity, Idempotence from p, q

Theorem

Let $(A, \wedge, \vee, \bot, \top, p, q)$ be a $d\ell pq$-algebra

1. *The operation $\cdot$ is commutative if and only if $p = q$.*
2. *If $p = q$ then $\cdot$ is associative if and only if*
$$p((px \wedge y) \vee (x \wedge py)) = (px \wedge py) \vee (x \wedge ppy).$$
3. *If $p = q$ and $x \leq px = ppx$ then $\cdot$ is associative if and only if*
$$px \wedge py \leq p((px \wedge y) \vee (x \wedge py)).$$
4. *The operation $\cdot$ is idempotent if and only if $x \leq px$ and $x \leq qx$, if and only if $p\top = \top = q\top$.*
5. *The operation $\cdot$ has an identity 1 if and only if $p1 = \top = q1$ and $(px \vee qx) \wedge 1 \leq x$.*
6. *If $\cdot$ has an identity then $\cdot$ is idempotent.*

Term equivalence for idemp. unary-determined BI-algebra

An operation p' is the **residual** of p if $px \leq y \iff x \leq p'y$ holds.

Theorem

1. *Let* $\mathbf{A}$ *be a Heyting algebra with an operation* p*, residual* p' *and constant* 1 *such that* $px \wedge py \leq p((px \wedge y) \vee (x \vee py))$*,* $x \leq px = ppx$*,* $p1 = \top$ *and* $px \wedge 1 \leq x$*.*

   Define $x{*}y = (px \wedge y) \vee (x \wedge py)$ *and* $x{-\!*}y = (px \to y) \wedge p'(x \wedge y)$*.*

   Then $(A, \wedge, \vee, \bot, \top, \to, *, -\!*, 1)$ *is an idempotent unary-determined BI-algebra.*

2. *Let* $(A, \wedge, \vee, \top, \bot, \to, *, -\!*, 1)$ *be an idempotent unary-determined BI-algebra, and define* $px = \top{*}x$ *and* $p'x = \top{-\!*}x$*.*

   Then $(A, \wedge, \vee, \to, \top, \bot, p, p', 1)$ *is a Heyting algebra with an operation* p *that has* p' *as residual and satisfies* $x \leq px = ppx$*,* $px \wedge py \leq p((px \wedge y) \vee (x \vee py))$*,* $p1 = \top$ *and* $px \wedge 1 \leq x$*.*

Distributive lattices (= Heyting algebras) of cardinality ≤ 6

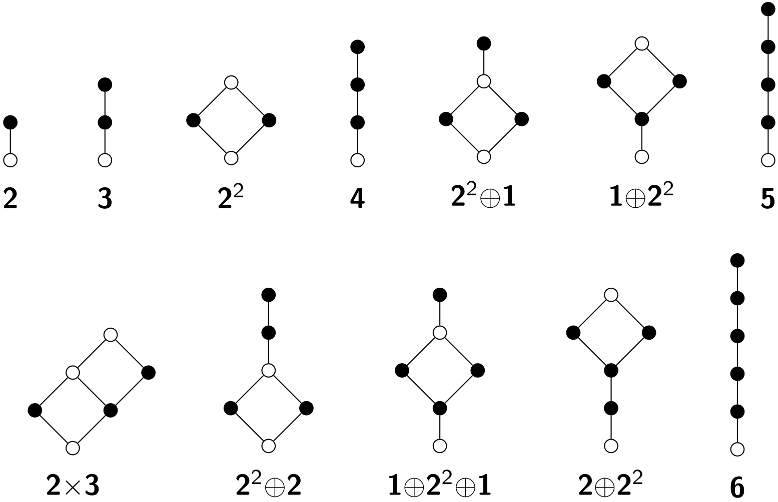

Figure: The completely join-irreducible elements are in black.

Downsets and completely join-irreducibles

Definition

Let $(W, \leq)$ be a poset. A **downset** is a subset $X \subseteq W$ such that $y \leq x \in X$ implies $y \in X$.

- Let $D(W, \leq)$ be the set of all downsets.
- The **downset lattice** is $(D(W, \leq), \cap, \cup, \emptyset, W)$.
- The downset lattice is a bounded distributive lattice.

Definition

An element x in a lattice A is **completely join-irreducible** if

$$x \neq \bigvee \{y \in A \mid y < x\}.$$

$J(A)$ denotes the set of completely join-irreducibles of A.

Kripke semantics for finite dℓ-magmas

Definition

$(J(A), \leq, R)$ is the **Birkhoff frame** of a finite $d\ell$-magma A with the ternary relation R defined by $R(x, y, z) \iff x \leq y{\cdot}z$.

- From $(x \vee y){\cdot}z = x{\cdot}z \vee y{\cdot}z$ it follows that $\cdot$ is order preserving.
- Hence R satisfies:

(R1) $u \leq x \;\&\; R(x, y, z) \implies R(u, y, z)$ (downward closure)
(R2) $R(x, y, z) \;\&\; y \leq v \implies R(x, v, z)$ (upward closure)
(R3) $R(x, y, z) \;\&\; z \leq w \implies R(x, y, w)$ (upward closure).

Definition

In general a **Birkhoff frame W** $= (W, \leq, R)$ is a poset $(W, \leq)$ with a ternary relation $R \subseteq W^3$ that satisfies (R1),(R2),(R3).

Birkhoff frames produce $d\ell$-magmas

Definition

For a Birkhoff frame $\mathbf{W}$ define the **downset algebra** $D(\mathbf{W}) = (D(W, \leq), \cap, \cup, \cdot, \emptyset, W)$, where for $Y, Z \in D(W, \leq)$

$$Y \cdot Z = \{x \in W \mid R(x, y, z) \text{ for some } y \in Y \text{ and } z \in Z\}.$$

$Y \cdot Z$ is a downset by (R1),(R2),(R3) of R.

Theorem

Let $\mathbf{W}$ *be a Birkhoff frame. Then*

- $D(\mathbf{W})$ *is a* $d\ell$*-magma.*
- $D(\mathbf{W})$ *is idempotent if and only if for all* $x, y, z \in W$, $R(x, x, x)$, *and* $(R(x, y, z) \implies x \leq y \text{ or } x \leq z)$.

PQ-Frames and P-Frames

Definition

$(W, \leq, P, Q)$ is a **PQ-frame** if

1. $(W, \leq)$ is a poset.
2. $u \leq x \ \& \ P(x, y) \ \& \ y \leq v \implies P(u, v)$
3. $u \leq x \ \& \ Q(x, y) \ \& \ y \leq v \implies Q(u, v)$

- A **P-frame** is a PQ-frame where $P = Q$.
- P is **reflexive** if $P(x, x)$ for all $x \in W$.
- P is **transitive** if $P(x, y) \ \& \ P(y, z) \implies P(x, z)$.

Correspondence theory for PQ-Frames and $d\ell pq$-algebras

Lemma

Let $\mathbf{W} = (W, \leq, P, Q)$ *be a PQ-frame, and* $\mathbf{A} = D(\mathbf{W})$ *a* $d\ell pq$*-algebra. If it exists, the constant* $1 \in A$ *corresponds to a downset* $E \subseteq W$*. Then*

1. $x \leq px$ *holds in* $\mathbf{A}$ *if and only if* P *is reflexive,*
2. $ppx \leq px$ *holds in* $\mathbf{A}$ *if and only if* P *is transitive,*
3. $px = qx$ *holds in* $\mathbf{A}$ *if and only if* $P = Q$*,*
4. $p1 = \top$ *holds in* $\mathbf{A}$ *if and only if* $\forall x \exists y (y \in E \ \& \ xPy)$ *holds in* $\mathbf{W}$*,*
5. $px \wedge 1 \leq x$ *holds in* $\mathbf{A}$ *if and only if* $x \in E \ \& \ xPy \Rightarrow x \leq y$ *in* $\mathbf{W}$*,*
6. $px \wedge py \leq p((px \wedge y) \vee (x \wedge py))$ *holds in* $\mathbf{A}$ *if and only if* $wPx \ \& \ wPy \Rightarrow \exists v (wPv \ \& \ (vPx \ \& \ v \leq y \text{ or } v \leq x \ \& \ vPy))$ *in* $\mathbf{W}$*.*

If $x \leq px = ppx$ then from (6) we get associativity in the term-equivalent $d\ell$-magma $(A, \wedge, \vee, \bot, \top, \cdot)$, where $x \cdot y = (px \wedge y) \vee (x \wedge py)$.

Preorder forest P-frames

A **preorder forest** P-**frame** is a P-frame such that P is a preorder (i. e. reflexive and transitive) and satisfies the formula

$$\text{(Pforest)} \quad xPy \text{ and } xPz \implies x \leq y \text{ or } x \leq z \text{ or } yPz \text{ or } zPy.$$

Theorem

Let $\mathbf{W} = (W, \leq, P)$ *be a preorder forest* P*-frame and* $D(\mathbf{W})$ *its corresponding downset algebra. Then*

- *the operation* $x*y = (px \wedge y) \vee (x \wedge py)$ *is associative in* $D(\mathbf{W})$,
- $E \subseteq W$ *is an identity element for* $*$ *in the downset algebra* $D(\mathbf{W})$ *if and only if* E *is a downset and* $pE = W$

  if and only if $(D(W, \leq), \cap, \cup, \emptyset, W, \to, *, -\!\!*, E)$ *is an idempotent unary-determined BI-algebra, where* $X -\!\!* Y = \{z \in W \mid X*\{z\} \subseteq Y\}$.

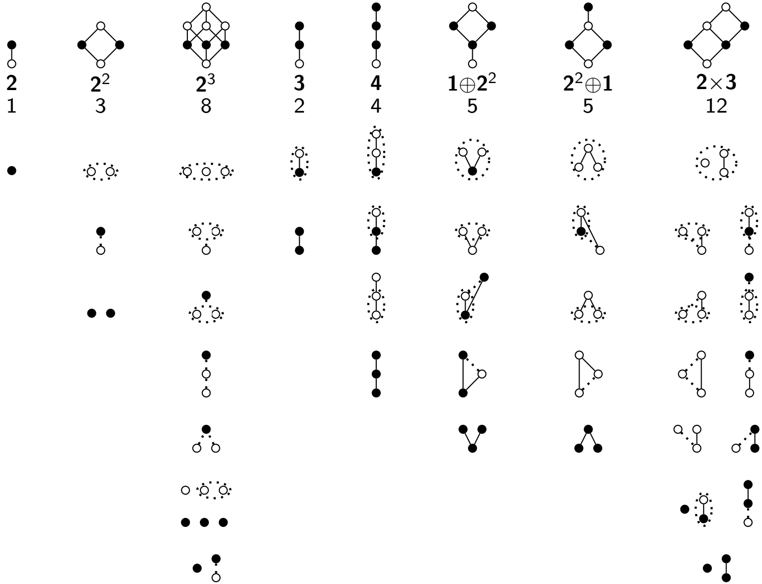

Figure: All 40 preorder forest P-frames $(W, \leq, P)$ with up to 3 join-irreducibles. Solid lines show $(W, \leq)$, dotted lines show the additional edges of P, and the identity (if it exists) is the set of black dots. The first row shows the lattice of downsets.

Conclusion

- Distributive lattices with unary operations are simpler than ones with binary operations. Hence the term-equivalence between unary-determined $d\ell$-magmas and $d\ell pq$-algebras is useful.
- We defined Birkhoff frames for $d\ell$-magmas, and PQ-frames for $d\ell pq$-algebras. These frames are logarithmic in size compared to the algebras.
- Preorder forest P-frames can be calculated more efficiently than idempotent unary-determined BI-algebras, and the P-frames can be drawn as Hasse diagrams of the poset (solid lines) and the preorder (dotted and solid lines).

n	1	2	3	4	5	6	7	8	9
all BI f_n	1	1	3	16	70	399	2261		
idem. BI f_n	1	1	2	6	15	44	115	326	
idem. u-d BI f_n	1	1	2	5	10	24	47	108	223

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