Decompositions of ordered algebraic structures

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Outline

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Definition

A *groupoid* is a set with a binary operation $x \cdot y$ (written xy)

It is *unital* if it has a constant e that is a unit (i.e., $xe = ex = x$)

A *monoid* is an associative unital groupoid

A *lattice-ordered* groupoid or *ℓ-groupoid* is a groupoid expanded with lattice operations $\vee, \wedge$ that satisfy the identities

$$x(y \vee z) = xy \vee xz \quad \text{and} \quad (x \vee y)z = xz \vee yz$$

Unital ℓ-groupoids, *ℓ-monoids* and *ℓ-groups* are defined similarly

They are *bounded* if there are constants $0, 1$ denoting the bottom and top element of the lattice reduct.

Mostly we consider *integral* bounded ℓ-groupoids (or *ibℓ*-groupoids for short), i.e. they have the top element 1 as the unit.

A *residuated* ℓ-groupoid (or $r\ell$-groupoid) is an ℓ-groupoid for which the residuals $\backslash, /$ exist relative to the groupoid operation, i.e.,

$$x \cdot y \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x\backslash z$$

An $ibr\ell$-monoid is also called an *FL_w-algebra*

An element c in an $ib\ell$-groupoid $\mathbf{A}$ is *complemented* if there exists $c' \in A$ such that $c \wedge c' = 0$ and $c \vee c' = 1$.

The *Boolean center* of $\mathbf{A}$ is the set $B(\mathbf{A})$ of all complemented elements.

The *direct decomposition* results below generalize similar results for MV-algebras [Cignoli, D'Ottaviano and Mundici 2000] and BL-algebras [Di Nola, Georgescu and Leustan 2000]. With the help of Prover9 [McCune 2008] it was shown that associativity is not needed for some of these results. The first lemma is essentially due to [Birkhoff 1967].

Lemma

Let $\mathbf{A}$ *be an ibℓ-groupoid and let* $c \in B(\mathbf{A})$*. Then* $x \wedge c = xc = cx$ *for all* $x \in A$*, and the Boolean center is a Boolean sublattice of central idempotent elements*

Proof.

Suppose A is an *ibℓ*-groupoid and $c \wedge d = 0$, $c \vee d = 1$. By integrality

$$xc \leq x \wedge c = (x \wedge c)(c \vee d) = (x \wedge c)c \vee (x \wedge c)d \leq xc \vee 0 = xc,$$

and similarly $cx \leq c \wedge x \leq cx$.

Suppose we also have $a \wedge b = 0$, $a \vee b = 1$. To see that $B(\mathbf{A})$ is a sublattice of $\mathbf{A}$, it suffices to show that $a \vee c$ and $b \wedge d$ are complements:

$(a \vee c) \wedge (b \wedge d) = (a \vee c)bd = abd \vee cbd = 0$ and

$(a \vee c) \vee (b \wedge d) = a \vee c \vee bd = a \vee c \vee bc \vee bd = a \vee c \vee b(c \vee d) = a \vee c \vee b = 1.$

Now $B(\mathbf{A})$ is complemented by definition, and it is a distributive lattice since $\cdot$ distributes over $\vee$, hence it is a Boolean lattice. □

Lemma

If $\mathbf{A}$ *is a residuated ib*ℓ*-groupoid then* $B(\mathbf{A})$ *is also closed under the residuals, the complement of* c *is* $-c = 0/c = c\backslash 0$ *and* $c\backslash x = x/c = -c \vee x$ *for all* $c \in B(\mathbf{A})$ *and* $x \in A$*.*

Proof.

For complements c, d and any $x \in A$ we have

$$c\backslash x = (c \vee d)(c\backslash x) = c(c\backslash x) \vee d(c\backslash x) \leq x \vee d$$

On the other hand $c(x \vee d) = cx \vee cd \leq x$ implies $x \vee d \leq c\backslash x$.

Hence $c\backslash x = d \vee x$, and for $x = 0$ we obtain $-c = c\backslash 0 = d$

Therefore $c\backslash x = -c \vee x$ for all $x \in A$

The results for $/$ follow similarly. □

For an $ib(r)\ell$-groupoid $\mathbf{A}$ and an element $c \in B(\mathbf{A})$, define

the *relativized subalgebra* $\mathbf{A}c = {\downarrow}c$ with unit $1^{\mathbf{A}c} = c$, operations $\wedge, \vee, \cdot$ restricted from $\mathbf{A}$,

and $\quad a\backslash b = (a\backslash^{\mathbf{A}} b) \wedge c, \quad a/b = (a/^{\mathbf{A}} b) \wedge c \quad$ for all $a, b \in {\downarrow}c$.

Lemma

For any $ib(r)\ell$-groupoid $\mathbf{A}$ and any $c \in B(\mathbf{A})$, the relativized subalgebra $\mathbf{A}c$ is an $ib(r)\ell$-groupoid

Proof.

By the first lemma, $x \wedge c = xc = cx$, so $\mathbf{A}c$ has c as a unit and is closed under $\wedge, \vee, \cdot$, hence it is an $ib\ell$-groupoid.
If $\mathbf{A}$ has residuals then for all $a, b, x \in \mathbf{A}c$ we have

$$ax \leq b \quad \text{iff} \quad x \leq^{\mathbf{A}} a\backslash^{\mathbf{A}} b \text{ and } x \leq^{\mathbf{A}} c,$$

whence $a\backslash b = (a\backslash^{\mathbf{A}} b) \wedge c$, and similarly $a/b = (a/^{\mathbf{A}} b) \wedge c$. □

Lemma

If $\mathbf{A}$ *is an ib(r)ℓ-monoid and* $c \in B(\mathbf{A})$ *then the map* $f : \mathbf{A} \to \mathbf{A}c$ *given by*

$$f(a) = ac \quad \text{is a surjective homomorphism,}$$

hence $\mathbf{A}c$ *satisfies all identities that hold in* $\mathbf{A}$*.*

Proof.

$f(1) = 1c = 1^{\mathbf{A}c}$, $\quad (a \wedge b)c = a \wedge b \wedge c = ac \wedge bc$ and

$(a \vee b)c = ac \vee bc \quad$ hence f preserves $\wedge, \vee$.

If $\cdot$ is associative then $(ab)c = abcc = (ac)(bc)$.

In any residuated lattice $x\backslash y \le xz\backslash yz$, hence $f(a\backslash^{\mathbf{A}} b) \le f(a)\backslash f(b)$.

For the opposite inequality, we have $ac(ac\backslash bc) \le bc \le b$

and therefore $c(ac\backslash bc) \le a\backslash b$. □

Theorem

If $\mathbf{A}$ *is an ib(r)ℓ-monoid and if* $c, d \in B(\mathbf{A})$ *are complements then*

$$\mathbf{A} \cong \mathbf{A}c \times \mathbf{A}d$$

Proof.

Consider the map $h : \mathbf{A} \to \mathbf{A}c \times \mathbf{A}d$ defined by $h(a) = (ac, ad)$.

The preceding lemma shows that h is a homomorphism, and

h has an inverse given by $(x, y) \mapsto x \vee y$

since $ac \vee ad = a(c \vee d) = a1 = a$ and

for $x \leq c$, $y \leq d$ we have

$$((x \vee y)c, (x \vee y)d) = (xc \vee yc, xd \vee yd) = (x, y)$$

□

Conversely, any direct decomposition of an $ib(r)\ell$-groupoid is obtained in this way, since the elements $(0,1)$, $(1,0)$ are complements.

Corollary

An $ib(r)\ell$-monoid is directly indecomposable iff its Boolean center contains only the elements $\{0,1\}$.

The structure of directly indecomposables can be further analyzed by using *subdirect decompositions*

However, we now consider a *poset product* that can even be used to decompose subdirectly irreducible algebras

The *poset product* uses a partial order on the index set to define a subset of the direct product.

Specifically, let $\mathbf{X} = (X, \leq)$ be a poset, and assume $\{\mathbf{A}_i : i \in X\}$ is a family of algebras that have two constant operations denoted $0, 1$.

The poset product of $\{A_i : i \in X\}$ is

$$\prod_{\mathbf{X}} A_i = \{f \in \prod_{i \in X} A_i : \quad f(i) = 0 \text{ or } f(j) = 1 \quad \text{for all } i < j \text{ in } X\}$$

If $\mathbf{X}$ is an *antichain* then the poset product is the same as the direct product

If $\mathbf{X}$ is a *chain* and the A_i are ordered, then the poset product is the (amalgamated) ordinal sum of the A_i

For an ℓ-groupoid $\mathbf{A}$ define $I_{\mathbf{A}} = \{c \in A : c \wedge x = cx = xc \text{ for all } x \in A\}$

Note that $\wedge$ distributes over $\vee$ in $I_{\mathbf{A}}$, but $I_{\mathbf{A}}$ need not be a subalgebra of $\mathbf{A}$

A *GBL-algebra* is a $r\ell$-monoid that satisfies
$x \le y \Rightarrow x = (x/y)y = y(y\backslash x)$

[J., Montagna 2006] prove that for bounded GBL-algebras, $\mathbf{I}_{\mathbf{A}}$ is a subalgebra, hence a Heyting algebra contained in $\mathbf{A}$,

and $B(\mathbf{A})$ is the subalgebra of complemented elements of $\mathbf{I}_{\mathbf{A}}$.

For MV-algebras $\mathbf{I}_{\mathbf{A}} = B(\mathbf{A})$

Lemma

Let $\mathbf{A}$ be an $ib(r)\ell$-monoid and let $a, b \in I_{\mathbf{A}}$ with $a \le b$.
Then the interval $[a, b] = \{x \in A : a \le x \le b\}$ is an $ib(r)\ell$-monoid, with
$0 = a$, $1 = b$,
$\wedge, \vee, \cdot$ *inherited from* $\mathbf{A}$, *and* $x\backslash y = (x\backslash^{\mathbf{A}} y) \wedge b$, $x/y = (x/^{\mathbf{A}} y) \wedge b$.
If $\mathbf{A}$ is a GBL-algebra, then so is $[a, b]$.

We now generalize the poset sum decomposition result of [J., Montagna 2006] from finite GBL-algebras to certain $ib(r)\ell$-monoids

Theorem

Consider an $ib(r)\ell$-monoid $\mathbf{A}$ *with a finite subalgebra* $\mathbf{C}$ *such that* $C \subseteq I_{\mathbf{A}}$,

and let $\mathbf{X}$ *be the dual of the partially ordered set of completely join irreducible elements of* $\mathbf{C}$.

For $c \in X$, *let* c_* *denote the unique lower cover of* c *in* $\mathbf{C}$.

If $\mathbf{A}c = \downarrow c_* \oplus [c_*, c]$ *for all* $c \in X$ *then* $\mathbf{A} \cong \prod_{\mathbf{X}} [c_*, c]$.

Corollary

Suppose $\mathbf{A}$ *is a normal GBL-algebra such that* $I_{\mathbf{A}}$ *is finite.*

Then $\mathbf{A}$ *is isomorphic to a poset product of GMV-algebras.*

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(Brief) Sage demo of posets package