Universal Algebra tutorial individualized exercise sheet #1

BLAST 2009, August 14

The algebra $\mathbf{A}$ below is a 4-element monoid with an additional unary operation f.

$\cdot$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2
f	2	1	0	3

If possible, calculate the following related structures and try to answer the questions.

(1) $(\mathsf{Sub}(\mathbf{A}), \cap, \vee)$

(2) $(\mathsf{Con}(\mathbf{A}), \cap, \vee)$

(3) $\mathbf{F}_{\mathsf{V}(\mathbf{A})}(2)$

(4) $(\mathsf{HS}(\mathbf{A}), \leq, \ll)$

(5) $\mathsf{Clo}_1(\mathbf{A})$

(6) $\mathsf{Pol}_1(\mathbf{A})$

(7) $(\mathsf{Aut}(\mathbf{A}), \circ, ^{-1}, \mathsf{id}_A)$

(8) $(\mathsf{End}(\mathbf{A}), \circ, \mathsf{id}_A)$

(9) Is $\mathbf{A}$ simple?

(10) Is $\mathbf{A}$ subdirectly irreducible?

(11) Find minimal subdirect decompositions for any subdirectly decomposable algebras in $\mathsf{HS}(\mathbf{A})$

(12) Is $\mathbf{A}$ congruence distributive? (How about $\mathsf{V}(\mathbf{A})$?)

(13) Is $\mathbf{A}$ congruence modular? (How about $\mathsf{V}(\mathbf{A})$?)

(14) Is $\mathbf{A}$ congruence meet semidistributive? (How about $\mathsf{V}(\mathbf{A})$?)

(15) Is $\mathbf{A}$ congruence join semidistributive? (How about $\mathsf{V}(\mathbf{A})$?)

(16) Is $\mathbf{A}$ congruence regular? (How about $\mathsf{V}(\mathbf{A})$?)

(17) Is $\mathbf{A}$ congruence uniform? (How about $\mathsf{V}(\mathbf{A})$?)

(18) Is $\mathbf{A}$ congruence permutable? (How about $\mathsf{V}(\mathbf{A})$?)

(19) Is $\mathbf{A}$ (semi/sub/demi/hemi/para/crypto/quasi/pre-)primal?

(20) Is $\mathbf{A}$ functionally complete or affine complete?

(21) Is $\mathbf{A}$ minimal?

(22) Calculate type labels for $\mathsf{Con}(\mathbf{A})$. (How about $\mathsf{types}(\mathsf{V}(\mathbf{A}))$?)

(23) Is $\mathbf{A}$ abelian? (How about $\mathsf{V}(\mathbf{A})$?)

(24) Is $\mathbf{A}$ strongly abelian? (How about $\mathsf{V}(\mathbf{A})$?)

(25) Is $\mathsf{V}(\mathbf{A})$ finitely based?

(26) Does $\mathsf{V}(\mathbf{A})$ have a decidable (quasi)equational or first-order theory?

Definitions

$\mathbf{B} \leq \mathbf{A}$ means $\mathbf{B}$ is isomorphic to a subalgebra of $\mathbf{A}$
$\mathbf{B} \ll \mathbf{A}$ means $\mathbf{A} \twoheadrightarrow \mathbf{B}$, i.e., $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$
For a class $\mathcal{K}$ of algebras $\mathsf{H}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$, $\mathsf{P}(\mathcal{K})$ are the classes of homomorphic images, subalgebras and products of members of $\mathcal{K}$. Also, $\mathsf{H}(\mathbf{A}) = \mathsf{H}(\{\mathbf{A}\})$, etc.
$\mathsf{V}(\mathcal{K}) = \mathsf{HSP}(\mathcal{K})$ is the smallest variety containing $\mathcal{K}$
$\mathsf{Sub}(\mathbf{A})$, $\mathsf{Con}(\mathbf{A})$ are the lattices of subalgebras and congruences of $\mathbf{A}$
$\mathsf{Clo}_n(\mathbf{A})$, $\mathsf{Pol}_n(\mathbf{A})$ are the n-ary term operations and polynomial operations on $\mathbf{A}$
$\mathsf{Aut}(\mathbf{A})$ and $\mathsf{End}(\mathbf{A})$ are the automorphism group and endomorphism monoid of $\mathbf{A}$
$\mathbf{A}$ is *simple* if it has exactly two congruences: id_A and A^2.
$\mathbf{A}$ is *subdirectly irreducible* if it has a unique smallest congruence above id_A.
$\mathbf{A}$ is *congruence distributive* if $\mathsf{Con}(\mathbf{A})$ is distributive, i.e., $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds.
$\mathbf{A}$ is *congruence modular* if $\mathsf{Con}(\mathbf{A})$ is modular, i.e. $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$ holds.
$\mathbf{A}$ is *congruence meet semidistributive* if $\mathsf{Con}(\mathbf{A})$ is meet semidistributive, i.e. $x \wedge y = x \wedge z \Rightarrow x \wedge (y \vee z) = x \wedge y$.
$\mathbf{A}$ is *congruence join semidistributive* if $\mathsf{Con}(\mathbf{A})$ is join semidistributive, i.e. $x \vee y = x \vee z \Rightarrow x \vee (y \wedge z) = x \vee y$.
$\mathbf{A}$ is *congruence regular* if each congruence of $\mathbf{A}$ is uniquely determined by one of its congruence classes.
$\mathbf{A}$ is *congruence uniform* if each $\theta \in \mathsf{Con}(\mathbf{A})$ is uniform, i.e., $|x/\theta| = |y/\theta|$ for all $x, y \in A$.
$\mathbf{A}$ is *congruence permutable* if $\theta \circ \psi = \psi \circ \theta$ for all $\theta, \psi \in \mathsf{Con}(\mathbf{A})$.
$\mathbf{A}$ is *primal* if every operation on A is a term operation.
$\mathbf{A}$ is *semiprimal* if every operation on A which preserves all subalgebras of $\mathbf{A}$ is a term operation.
$\mathbf{A}$ is *subprimal* if it is semiprimal and has precisely one proper subalgebra.
$\mathbf{A}$ is *demiprimal* if it has no proper subalgebras and every operation on A which preserves all automorphisms of $\mathbf{A}$ is a term operation.
$\mathbf{A}$ is *hemiprimal* if every operation on A which preserves all congruences of $\mathbf{A}$ is a term operation.
$\mathbf{A}$ is *paraprimal* if every subalgebra of $\mathbf{A}$ is simple and $\mathsf{V}(\mathbf{A})$ is congruence permutable.
$\mathbf{A}$ is *cryptoprimal* if it is simple, has no proper subalgebra and $\mathsf{V}(\mathbf{A})$ is congruence distributive.
The *ternary discriminator* $t : A^3 \to A$ is defined by $t(x, y, z) = z$ if $x = y$ and $t(x, y, z) = x$ otherwise.
$\mathbf{A}$ is *quasiprimal* if the ternary discriminator t is a term operation of $\mathbf{A}$.
$\mathbf{A}$ is *preprimal* if $\mathsf{Clo}(\mathbf{A})$ is maximal proper in the lattice of all clones on A.
$\mathbf{A}$ is *functionally complete* if every operation on A is a polynomial operation.
$\mathbf{A}$ is *affine complete* if every operation on A which preserves all congruences of $\mathbf{A}$ is a polynomial operation.
$\mathbf{A}$ is *minimal* if it is finite, $|A| \geq 2$ and every unary polynomial of $\mathbf{A}$ is a constant operation or a permutation on A.
$\mathbf{A}$ is of *type* 1 or *unary type* if it is polynomially equivalent to a $\mathbf{G}$-set.
$\mathbf{A}$ is of *type* 2 or *affine type* if it is polynomially equivalent to a vector space.
$\mathbf{A}$ is of *type* 3 or *Boolean type* if it is polynomially equivalent to the 2-element Boolean algebra.
$\mathbf{A}$ is of *type* 4 or *lattice type* if it is polynomially equivalent to the 2-element lattice.
$\mathbf{A}$ is of *type* 5 or *semilattice type* if it is polynomially equivalent to the the 2-element semilattice.
$\mathbf{A}$ is *abelian* if $\mathbf{A}$ satisfies $t(u, y_1, \ldots, y_n) = t(u, z_1, \ldots, z_n) \Rightarrow t(v, y_1, \ldots, y_n) = t(v, z_1, \ldots, z_n)$ for all terms t.
$\mathbf{A}$ is *strongly abelian* if $\mathbf{A}$ satisfies $t(u, y_1, \ldots, y_n) = t(v, z_1, \ldots, z_n) \Rightarrow t(w, y_1, \ldots, y_n) = t(w, z_1, ..., z_n)$ for all terms t.
A variety $\mathcal{V}$ is *finitely based* if $\mathcal{V} = \mathsf{Mod}(E)$ for a finite set of equations E.
$\mathcal{V}$ has a decidable (quasi)equational theory if there is an algorithm that decides which (quasi)equations hold in $\mathcal{V}$.
$\mathcal{V}$ has a decidable first-order theory if there is an algorithm that decides which first-order formulas hold in $\mathcal{V}$.