

A GUIDE FOR MORTALS TO TAME CONGRUENCE THEORY

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Tame congruence theory is not an easy subject and it takes a considerable amount of effort to understand it. When I started this project, I believed that this theory can be better explained than the standard text: D. Hobby and R. McKenzie’s “The Structure of Finite Algebras.” I followed a more gradual approach in introducing new concepts and immediately describing the structure of finite algebras at that level. The reader is encouraged to consult the book (TCT references) while reading this guide.

1. MINIMAL ALGEBRAS

In this section we characterize the minimal algebras, up to polynomial equivalence. It will turn out that they can be nicely split into 5 distinct types.

Definition 1.1. [TCT 2.14] An algebra $\mathbf{M}$ is *minimal* if it is finite, $|M| \geq 2$, and

$$\text{Pol}_1 \mathbf{M} \subseteq \{ \text{constants} \} \cup \{ \text{permutations on } M \}.$$

Note that by “constants” here we mean unary operations whose ranges are one element subsets of M .

Exercise 1.2. Prove the following statements:

- (1) every two-element algebra is minimal;
- (2) every finite nontrivial vectorspace is minimal;
- (3) a group $\mathbf{G}$ is minimal iff $\mathbf{G} \cong \mathbb{Z}_p^n$ for some prime p and $n \geq 1$;
- (4) an algebra with a semilattice operation is minimal iff it has only two elements.

Definition 1.3. [TCT 4.3] Let f be an n -ary operation and $i < n$. We define an operation $f_{(i)}^k$ by induction on $k \geq 0$. Put $f_{(i)}^0(x_0, \dots, x_{n-1}) = x_i$, and

$$f_{(i)}^{k+1}(x_0, \dots, x_{n-1}) = f(x_0, \dots, x_{i-1}, f_{(i)}^k(x_0, \dots, x_{n-1}), x_{i+1}, \dots, x_{n-1}).$$

This process is called *iterating f in its i -th coordinate*.

Lemma 1.4. [TCT 4.4] *Let A be a finite set. Then there exists $k > 0$ such that for all operation $f : A^n \rightarrow A$, elements $a_0, \dots, a_{n-1} \in A$, and $i < n$ the following hold.*

- (1) $f_{(i)}^k(\bar{x}) = f_{(i)}^k(x_0, \dots, x_{i-1}, f_{(i)}^k(\bar{x}), x_{i+1}, \dots, x_{n-1})$.
- (2) *If the unary polynomial $g(x) = f(a_0, \dots, a_{i-1}, x, a_{i+1}, \dots, a_{n-1})$ is a permutation, then $f_{(i)}^k(a_0, \dots, a_{i-1}, x, a_{i+1}, \dots, a_{n-1}) = x$.*

Proof. Put $k = |A|!$. By the definition of $f_{(i)}^k$, $g^k(x) = g(g^{k-1}(x)) = f_{(i)}^k(a_0, \dots, a_{i-1}, x, a_{i+1}, \dots, a_{n-1})$. Since A is finite, the sequence $a_i, g(a_i), g^2(a_i), \dots, g^k(a_i)$ must contain a repetition, a cycle, whose order divides k . Therefore, $g^k(a_i) = g^{2k}(a_i) = g^k(g^k(a_i))$. This holds for arbitrary elements $a_0, \dots, a_{n-1} \in A$, which proves (1). Finally, if g is a permutation, then $g^k = \text{id}$, which proves (2). $\square$

Definition 1.5. Let $f : A_0 \times \dots \times A_{n-1} \rightarrow A$ be a function. We say that f depends on its i -th coordinate ($i < n$) if there exists $a_j \in A_j$ for all $j \neq i$ such that the unary function $f(a_0, \dots, a_{i-1}, x, a_{i+1}, \dots, a_{n-1})$ is not constant.

Lemma 1.6 (A. Salomaa). [TCT 4.1] *Let $f : A_0 \times \dots \times A_{n-1} \rightarrow A$ depends on all its variables, where $n \geq 2$. Then there exist $i, j < n$, $i \neq j$, and $a_i \in A_i, a_j \in A_j$ such that $f(x_0, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_{n-1})$ and $f(x_0, \dots, x_{j-1}, a_j, x_{j+1}, \dots, x_{n-1})$ depend on all their variables.*

Proof. For any $i < n$ and $a \in A_i$, let $D(a, i)$ denote the set of indices of the variables which $f(x_0, \dots, x_{i-1}, a, x_{i+1}, \dots, x_{n-1})$ depends on.

CLAIM. *If $j \neq i$ and $j \notin D(a, i)$ then $D(a, i) \subset D(b, j)$ for some $b \in A_j$.*

Since f depends on all its variables, there exists $b \in A_j$ such that $i \in D(b, j)$. For this choice, it is easy to see that $D(a, i) \subset D(b, j)$.

We can choose $i < n$ and $a \in A_i$ such that $D(a, i)$ is maximal under inclusion. By the claim, $|D(a, i)| = n - 1$. Since f depends on the i th coordinate, there is $k < n$ and $c \in A_k$ such that $i \in D(c, k)$. By choosing $D(b, j)$ maximal above $D(c, k)$ we get that $|D(b, j)| = n - 1$ and $i \in D(b, j)$. Therefore $i \neq j$. $\square$

Corollary 1.7. [TCT 4.2] *Let $\mathbf{A}$ be an algebra, and suppose that $\mathbf{A}$ has a polynomial that depends on at least n variables. Then for all $k \leq n$, $\mathbf{A}$ has a polynomial of k variables that depends on all of its variables.* $\square$

Definition 1.8. Two algebras are *polynomially equivalent* if they have the same universe and the same clone of polynomial operations.

Theorem 1.9 (P.P. Pálffy). [TCT 4.7, 4.6] *Every minimal algebra $\mathbf{M}$ with $|M| \geq 3$ and having a polynomial operation which depends on more than one variable, is polynomially equivalent with a vector space.*

Proof. First we explore the consequences of $\mathbf{M}$ being minimal and having at least 3 elements.

CLAIM 1. *Every binary polynomial p satisfies the term-condition:*

$$p(u, a) = p(u, b) \implies p(v, a) = p(v, b).$$

Assume that $p(u, a) = p(u, b)$ and $p(v, a) \neq p(v, b)$, and we want to get a contradiction. Clearly $u \neq v$ and $a \neq b$. By Lemma 1.4 we can choose $k > 0$ such that $q(x, y) = p_{(1)}^k(x, y)$ satisfies $q(x, q(x, y)) = q(x, y)$. Since $p(v, a) \neq p(v, b)$, the unary polynomial $p(v, y)$ in y is a permutation. Therefore $q(v, y) = y$. Since $p(u, a) = p(u, b)$, the unary polynomial $p(u, y)$ in y is a constant. Therefore $q(u, y) = c$ for some element $c \in M$. Since $q(u, c) = q(v, c) = c$, the unary polynomial $q(x, c)$ in x is constant and equal to c . Now take $w \in M - \{u, v\}$ and $d \in M - \{c\}$. If $p(w, y)$ in y is a permutation then $q(w, y) = y$. Thus $q(u, d) = c$ and $q(v, d) = q(w, d) = d$. Hence the unary polynomial $q(x, d)$ in x is neither a permutation nor a constant, which is a contradiction. On the other hand if $p(w, y)$ in y is a constant then $q(w, y)$ is constant and equal to $q(w, c) = c$. Thus $q(u, d) = q(w, d) = c$ and $q(v, d) = d$. Hence the unary polynomial $q(x, d)$ in x is neither a permutation nor a constant, which is a contradiction once again.

Now we extend the previous claim to polynomials of higher arity. Note that this implication is still *not* the term condition.

CLAIM 2. *If $p \in \text{Pol}_{n+1} \mathbf{M}$, $\bar{u}, \bar{v} \in M^n$, and $a, b \in M$ then*

$$p(\bar{u}, a) = p(\bar{u}, b) \implies p(\bar{v}, a) = p(\bar{v}, b).$$

This is a standard argument. By the previous claim we can replace each variable of $\bar{u}$ by the corresponding variable of $\bar{v}$ one by one, and keep the sides equal.

Now we are ready to use the hypothesis that $\mathbf{M}$ has a polynomial which depends on more than one variables.

CLAIM 3. *$\mathbf{M}$ has a Mal'cev polynomial d .*

Let $p \in \text{Pol} \mathbf{M}$ be a polynomial which depends on more than one variables. By Corollary 1.7 we can assume that p is binary. Now we show that $p(x, y)$ is a quasigroup operation. Since $p(x, y)$ depends on

y , there exist $v, a, b \in M$ such that $p(v, a) \neq p(v, b)$. Then by Claim 1, $p(u, a) \neq p(u, b)$ for all $u \in M$. Hence $p(u, y)$, as a function of y , is a permutation for each u . Similarly, $p(x, u)$, as a function of x , is a permutation for each $u \in M$.

Now by Lemma 1.4 (2) we have polynomials $f(x, y) = p_{(0)}^{k-1}(x, y)$ and $g(x, y) = p_{(1)}^{k-1}(x, y)$ such that $p(f(x, y), y) = x$ and $p(x, g(x, y)) = y$. Put

$$d(x, y, z) = p(f(x, g(y, y)), g(y, z)).$$

Since $p(y, g(y, y)) = y$ and $p(f(y, g(y, y)), g(y, y)) = y$, and $p(x, g(y, y))$, as a function of x , is a permutation, we have that $y = f(y, g(y, y))$. Thus $d(y, y, z) = p(y, g(y, z)) = z$. On the other hand, $d(x, y, y) = p(f(x, g(y, y)), g(y, y)) = x$.

We are going to define a vector space on M using the Mal'cev operation $d(x, y, z)$, which will turn out to be the same as $x - y + z$. Pick an arbitrary element of M , call it 0, and define $x + y = d(x, 0, y)$ and $-x = d(0, x, 0)$.

CLAIM 4. *The algebra $\langle M; +, -, 0 \rangle$ is an Abelian group.*

To prove the claim we define the following polynomials and apply Claim 2 to them:

$$\begin{aligned} d_1(x, y, z, a) &= d(d(x, 0, a), 0, d(y, a, z)), \\ d_2(x, a) &= d(x, a, d(a, x, 0)), \\ d_3(x, y, a) &= d(a, 0, d(x, a, y)). \end{aligned}$$

Notice that $d_1(0, y, 0, 0) = y = d_1(0, y, 0, y)$, hence $x + (y + z) = d_1(x, y, z, 0) = d_1(x, y, z, y) = (x + y) + z$. Similarly, $d_2(0, 0) = d_2(0, x)$, hence $x + (-x) = d_2(x, 0) = d_2(x, x) = 0$. And again, $d_3(0, 0, 0) = d_3(0, 0, y)$, hence $x + y = d_3(x, y, 0) = d_3(x, y, y) = y + x$. Finally, $x + 0 = d(x, 0, 0) = x$, which concludes the proof of the claim.

CLAIM 5. *For each $p \in \text{Pol}_n \mathbf{M}$,*

$$p(x_0, \dots, x_{n-1}) = \sum_{i=0}^{n-1} p_i(x_i) - (n-1)p(0, \dots, 0),$$

where $p_i(x_i) = p(0, \dots, 0, x_i, 0, \dots, 0)$.

We prove it by induction. For $n = 1$ the statement is trivial. For $n = 2$, consider the polynomial $f(x, y) = p(x, y) - p(0, y)$. Notice that $f(0, y) = f(0, 0)$, hence $p(x, y) - p(0, y) = f(x, y) = f(x, 0) =$

$p(x, 0) - p(0, 0)$. Now let us assume that the claim holds for $n \geq 2$, and prove it for $n + 1$. Our assumption gives

$$\begin{aligned} p(x_0, \dots, x_{n-1}, y) = \\ p(x_0, 0, \dots, 0, y) + \dots + p(0, \dots, 0, x_{n-1}, y) - (n-1)p(0, \dots, 0, y). \end{aligned}$$

Now using the equation

$$\begin{aligned} p(0, \dots, 0, x_i, 0, \dots, 0, y) = \\ p(0, \dots, 0, x_i, 0, \dots, 0, 0) + p(0, \dots, 0, y) - p(0, \dots, 0) \end{aligned}$$

we get the desired conclusion.

CLAIM 6. *Let $F = \{\alpha \in \text{Pol}_1 \mathbf{M} : \alpha(0) = 0\}$. Then $\langle F; +, \circ \rangle$ is a field where $(\alpha + \beta)(x) = \alpha(x) + \beta(x)$ and $\circ$ is the composition operation.*

Clearly, $\langle F; \circ, \text{id} \rangle$ is a monoid and $\langle F; +, 0 \rangle$ is an Abelian group where 0 is the constant 0-valued function. By applying Claim 5 to $\alpha(x + y)$ we see that $\alpha(x + y) = \alpha(x + 0) + \alpha(0 + y) - \alpha(0 + 0)$, that is, $\alpha(x + y) = \alpha(x) + \alpha(y)$. This shows that $\alpha \circ (\beta + \gamma) = \alpha \circ \beta + \alpha \circ \gamma$ and $(\beta + \gamma) \circ \alpha = \beta \circ \alpha + \gamma \circ \alpha$, hence $\langle F; +, \circ \rangle$ is a ring. By the minimality of $\mathbf{M}$ if $\alpha \neq 0$ then $\alpha^k = \text{id}$ for some $k > 0$, hence $\langle F; +, \circ \rangle$ is a division ring. Since M is finite, F is finite, and therefore $\langle F; +, \circ \rangle$ is a finite field.

By defining $\alpha \cdot x = \alpha(x)$ for $\alpha \in F$ and $x \in M$, we clearly have a vector space $\mathbf{V} = \langle M; +, \alpha \cdot x (\alpha \in F) \rangle$. The operations of $\mathbf{V}$ belong to $\text{Pol} \mathbf{M}$. On the other hand, Claim 5 shows that each polynomial operation $p \in \text{Pol}_n \mathbf{M}$ is expressible as

$$p(x_0, \dots, x_{n-1}) = \sum_{i=0}^{n-1} \alpha_i \cdot x_i + c,$$

where $\alpha_i(x) = p_i(x) - p_i(0)$ and $c = p(0, \dots, 0)$. This concludes our proof. $\square$

By Pálffy's theorem we can easily characterize the minimal algebras of at least three elements. On the one hand we have the algebras polynomially equivalent with finite vector spaces, and on the other the algebras in which each operation depends on at most one variable. If $\mathbf{M}$ is such an algebra then each $p \in \text{Pol}_n \mathbf{M}$ is either a constant or $p(x_0, \dots, x_{n-1}) = \alpha(x_i)$ for some permutation α and variable x_i . Hence $\mathbf{M}$ is polynomially equivalent to a G -set, a permutation group acting on a set. What remains to be described are the two element minimal algebras.

Lemma 1.10. [TCT 4.8] *Every algebra $\mathbf{M}$ on a two element set $\{0, 1\}$ is polynomially equivalent with one of the following:*

$$\begin{aligned} \mathbf{E}_0 &= \langle \{0, 1\} \rangle, & \mathbf{E}_1 &= \langle \{0, 1\}; ' \rangle, & \mathbf{E}_2 &= \langle \{0, 1\}; + \rangle, \\ \mathbf{E}_3 &= \langle \{0, 1\}; \vee, \wedge, ' \rangle, & \mathbf{E}_4 &= \langle \{0, 1\}; \vee, \wedge \rangle, \\ \mathbf{E}_5 &= \langle \{0, 1\}; \vee \rangle, & \mathbf{E}_6 &= \langle \{0, 1\}; \wedge \rangle, \end{aligned}$$

where $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x + y = (x + y) \bmod 2$, and $x' = 1 - x$.

Proof. If every operation is essentially unary (does not depend on more than one variable) then $\mathbf{M}$ is polynomially equivalent to either $\mathbf{E}_0$ or $\mathbf{E}_1$.

In Pálffy's proof we needed that $|M| \geq 3$ only to show Claim 1. Thus if all binary polynomials of $\mathbf{M}$ satisfy the term-condition then $\mathbf{M}$ is polynomially equivalent with a two element vector space, i.e., with $\mathbf{E}_2$.

So pick $f \in \text{Pol}_2 \mathbf{M}$ which does not satisfy the term-condition. This essentially rules out that there are two 0's and two 1's in the operation table of f . Indeed, if the two 1's are in the same row or column, then f is essentially unary. If the two 1's are in diagonal then $f(x, y) = x + y$ or $f(x, y) = (x + y)'$. In either case f satisfies the term-condition. So either there are three 1's or a single 1 in the operation table of f .

If the operation $'$ is a polynomial of $\mathbf{M}$, then an appropriate combination of $f(x, y)$ and $'$ gives us $\vee$ and $\wedge$. Hence $\mathbf{M}$ is polynomially equivalent with $\mathbf{E}_3$, the full clone on $\{0, 1\}$.

So assume that the operation $'$ is not a polynomial of $\mathbf{M}$. This means that each operation of $\mathbf{M}$ is monotone. This essentially gives us only two possibilities for f , one is $\vee$ and the other is $\wedge$. Without loss of generality we can assume that $f = \wedge$.

Assume that $\mathbf{M}$ is not polynomially equivalent with $\mathbf{E}_6$. thus we can take $g \in \text{Pol}_n \mathbf{M} - \text{Pol}_n \langle \{0, 1\}; \wedge \rangle$. Consider the inverse image $G = g^{-1}(\{1\}) \subseteq 2^n$. Since g is order preserving, G is a filter of the lattice 2^n . Note that if G has a least element $a \in 2^n$, then $g(\bar{x}) = \bigwedge \{x_i : a_i = 1\} \in \text{Pol}_n \langle \{0, 1\}; \wedge \rangle$. So G must have at least two minimal elements a and b . Clearly, $g(a \wedge b) = 0$. Now take the polynomial $g(z_0, \dots, z_{n-1})$ and replace z_i by 1 if $a_i = b_i = 1$, by x if $a_i = 1$ and $b_i = 0$, by y if $a_i = 0$ and $b_i = 1$, and by 0 if $a_i = b_i = 0$. The result is a binary polynomial $h(x, y)$ for which $h(0, 0) = g(a \wedge b) = 0$, $h(1, 0) = g(a) = 1$, $h(0, 1) = g(b) = 1$, and $h(1, 1) = g(a \vee b) = 1$. This shows that $h = \vee$, hence $\text{Pol } \mathbf{M} \supseteq \text{Pol } \mathbf{E}_4$.

Finally, it is easy to see that every order preserving operation on $\{0, 1\}$ is a polynomial of $\langle\{0, 1\}, \vee, \wedge\rangle$. Thus $\text{Pol } \mathbf{M} = \text{Pol } \mathbf{E}_4$. $\square$

Now we can fully characterize the minimal algebras. The algebras $\mathbf{E}_0$, $\mathbf{E}_1$ are G -sets, and $\mathbf{E}_2$ is a vector space. They nicely extend the previous two classes of minimal algebras of at least three elements. The algebras $\mathbf{E}_3$, $\mathbf{E}_4$, and $\mathbf{E}_5$ will represent new classes. Note that although $\mathbf{E}_6$ is isomorphic to $\mathbf{E}_5$, they are not polynomially equivalent.

Definition 1.11. [TCT 4.10] Let $\mathbf{M}$ be a finite algebra.

- (1) $\mathbf{M}$ is of type **1**, or *unary type*, if polynomially equivalent to a G -set.
- (2) $\mathbf{M}$ is of type **2**, or *affine type*, if polynomially equivalent to a vector space.
- (3) $\mathbf{M}$ is of type **3**, or *boolean type*, if polynomially equivalent to a two element Boolean algebra.
- (4) $\mathbf{M}$ is of type **4**, or *lattice type*, if polynomially equivalent to a two element lattice.
- (5) $\mathbf{M}$ is of type **5**, or *semilattice type*, if polynomially equivalent to a two element semilattice.

Corollary 1.12. [TCT 4.11] *An algebra $\mathbf{M}$ is minimal iff it is of one of the types 1–5.* $\square$

2. $\langle\delta, \vartheta\rangle$ -MINIMAL ALGEBRAS

In this section we extend our five-fold classification to a broader class of algebras. As you can recall, an algebra $\mathbf{M}$ is minimal if every unary polynomial p of $\mathbf{M}$ is either a permutation or a constant. One can see that p being a constant is equivalent to the—unnatural—condition that $p(1_{\mathbf{M}}) \subseteq 0_{\mathbf{M}}$, where $0_{\mathbf{M}}, 1_{\mathbf{M}} \in \text{Con } \mathbf{M}$. But this condition naturally extends to other congruence quotients.

By a *congruence quotient* in an algebra $\mathbf{C}$ we simply mean a pair $\langle\delta, \vartheta\rangle$ of congruences of $\mathbf{C}$ with $\delta < \vartheta$.

Definition 2.1. [TCT 2.13] Let $\langle\delta, \vartheta\rangle$ be a congruence quotient in an algebra $\mathbf{C}$. We say that a polynomial $p \in \text{Pol}_1 \mathbf{C}$ *collapses ϑ into δ* if $p(\vartheta) \subseteq \delta$. The algebra $\mathbf{C}$ is called *minimal relative to $\langle\delta, \vartheta\rangle$* , or *$\langle\delta, \vartheta\rangle$ -minimal*, if it is finite, $|C| \geq 2$, and

$$\text{Pol}_1 \mathbf{C} \subseteq \{ \text{operations collapsing } \vartheta \text{ into } \delta \} \cup \{ \text{permutations on } C \}.$$

Definition 2.2. [TCT 2.15] Let $\mathbf{C}$ be an $\langle \delta, \vartheta \rangle$ -minimal algebra. By a $\langle \delta, \vartheta \rangle$ -trace we mean a ϑ -class, as a subset of C , which contains at least two δ -classes. The set of $\langle \delta, \vartheta \rangle$ -traces is denoted by $N_{\mathbf{C}}(\delta, \vartheta)$. The *body* of $\mathbf{C}$ with respect to $\langle \delta, \vartheta \rangle$ is defined to be the union of traces. The *tail* of $\mathbf{C}$ with respect to $\langle \delta, \vartheta \rangle$ is the complement of the body in C .

Lemma 2.3. *Let $\langle \delta, \vartheta \rangle$ be a congruence pair in an algebra $\mathbf{C}$. If $\mathbf{C}$ is $\langle \delta, \vartheta \rangle$ -minimal, then $\mathbf{C}/\delta$ is $\langle 0, \vartheta/\delta \rangle$ -minimal. Moreover, if they are minimal then*

$$N_{\mathbf{C}/\delta}(0, \vartheta/\delta) = \{ N/\delta|_N : N \in N_{\mathbf{C}}(\delta, \vartheta) \}.$$

Exercise 2.4. Show that the converse of Lemma 2.3 does not hold, that is, construct an algebra $\mathbf{C}$ and a congruence pair $\langle \delta, \vartheta \rangle$ such that $\mathbf{C}/\delta$ is $\langle 0, \vartheta/\delta \rangle$ -minimal but $\mathbf{C}$ is not $\langle \delta, \vartheta \rangle$ -minimal.

The following lemma gives us the connection between $\langle \delta, \vartheta \rangle$ -minimal and minimal algebras.

Lemma 2.5. *Let $\mathbf{C}$ be an $\langle \delta, \vartheta \rangle$ -minimal algebra, and N be a $\langle \delta, \vartheta \rangle$ -trace. Then the algebra $\mathbf{C}|_N/\delta|_N$ is minimal.*

Proof. Note that

$$\text{Pol } \mathbf{C}|_N/\delta|_N = \{ p|_N/\delta|_N : p \in \text{Pol}_n \mathbf{C} \text{ and } p(N^n) \subseteq N \}.$$

So take an arbitrary unary polynomial $p|_N/\delta|_N \in \text{Pol}_1 \mathbf{C}|_N/\delta|_N$. If p is a permutation of C , then since $p(N) \subseteq N$, $p|_N$ is a permutation of N and $p|_N/\delta|_N$ is a permutation of $N/\delta|_N$. On the other hand, if p collapses ϑ into δ , then $p(N)$ is contained in a single block of $\delta|_N$, hence $p|_N/\delta|_N$ is constant on $N/\delta|_N$. $\square$

Notice that the minimal algebra $\mathbf{C}|_N/\delta|_N$, which corresponds to the $\langle \delta, \vartheta \rangle$ -trace N , is the same as the minimal algebra $(\mathbf{C}/\delta)|_{N/\delta}$, which corresponds to the $\langle 0, \vartheta/\delta \rangle$ -trace $N/\delta|_N$. By the type of N we simply mean the type of the corresponding minimal algebra.

Lemma 2.6. [TCT 4.12, 4.15] *Let $\mathbf{C}$ be a $\langle \delta, \vartheta \rangle$ -minimal algebra, and N be a $\langle \delta, \vartheta \rangle$ -trace of type **3**, **4** or **5**. Then*

- (1) N is the disjoint union of two δ -classes I and O , one of which, say I , contains only a single element $1 \in C$.
- (2) There is a binary polynomial h such that $h(x, x) = h(1, x) = h(x, 1) = x$, $h(O, O) \subseteq O$, and $h(x, h(x, y)) = h(x, y)$.
- (3) N is the only $\langle \delta, \vartheta \rangle$ -trace in $\mathbf{C}$.
- (4) If $x \in C - \{1\}$ and $c \in O$ then $h(x, c) \stackrel{\delta}{\equiv} h(c, x) \stackrel{\delta}{\equiv} x$.

Proof. We know that N is a (disjoint) union of two δ -classes I and O , and $\mathbf{C}|_N/\delta|_N$ has a semilattice operation $p|_N/\delta|_N$. We can assume that $p(O, I) \subseteq O$, hence $p(I, O) \subseteq O$, $p(I, I) \subseteq I$, and $p(O, O) \subseteq O$.

Since $p(I, I) \subseteq I$ and $p(O, O) \subseteq O$, the polynomial $f(x) = p(x, x)$ cannot collapse ϑ into δ , hence it is a permutation on C . Choose $k > 0$ such that $f^k(x) = x$, and define $q(x, y) = f^{k-1}p(x, y)$. Clearly $q(x, x) = x$, and q behaves the same way as p with respect to I and O .

By Lemma 1.4 there exists $l > 0$ such that the function $g(x, y) = q_{(0)}^l(x, y)$ satisfies $g(g(x, y), y) = g(x, y)$. Note that $g(x, x) = x$, and g behaves the same way as p with respect to I and O . If $a \in I$, then $q(a, I) \subseteq I$ and $q(a, O) \subseteq O$. Therefore $q(a, y)$ is a permutation, and $g(a, y) = y$.

Applying Lemma 1.4 once more we can choose $m > 0$ such that the function $h(x, y) = g_{(1)}^m(x, y)$ satisfies $h(x, h(x, y)) = h(x, y)$. Note that $h(x, x) = x$, and h behaves the same way as p with respect to I and O . If $a \in I$, then $g(I, a) \subseteq I$ and $g(O, a) \subseteq O$. Therefore $g(x, a)$ is a permutation, and $h(x, a) = x$. On the other hand $h(a, y) = y$ because $g(a, y)$ is a permutation. Now if $a, b \in I$, then $a = h(a, b) = b$, which proves (1). Applying our knowledge of $I = \{1\}$ shows (2).

To prove (3) take a pair $\langle a, b \rangle \in \vartheta - \delta$, and an element $c \in O$. Since $h(I \cup O, O) \subseteq O$, the polynomial $h(x, c)$ collapses ϑ into δ , thus $\langle h(a, c), h(b, c) \rangle \in \delta$. We can assume that $\langle a, h(a, c) \rangle \notin \delta$. But $a = h(a, 1)$, thus $\langle h(a, 1), h(a, c) \rangle \notin \delta$ and therefore the polynomial $h(a, x)$ is a permutation. Since $h(a, a) = h(a, 1)$, we conclude that $a = 1$ and $a, b \in N$.

Statement (4) is a consequence of the previous ones. If $x \in O$, then it is clear from $h(O, O) \subseteq O$. If $x \notin O$, then x is in the tail, and $h(x, c) \vartheta h(x, 1) = x$ implies that $h(x, c) \delta x$. $\square$

Corollary 2.7. [TCT 4.17] *Let $\mathbf{C}$ be a $\langle \delta, \vartheta \rangle$ -minimal algebra, and N be a $\langle \delta, \vartheta \rangle$ -trace of type **3** or **4**. Then*

- (1) $N = \{0, 1\}$ for some pair $\langle 0, 1 \rangle \in \vartheta - \delta$.
- (2) *There are binary polynomials f, g such that $f(x, x) = f(1, x) = f(x, 1) = g(x, x) = g(0, x) = g(x, 0) = x$ for all $x \in C$. Moreover, $f(x, f(x, y)) = f(x, y)$, and $g(x, g(x, y)) = g(x, y)$.*
- (3) N is the only $\langle \delta, \vartheta \rangle$ -trace in $\mathbf{C}$.
- (4) *For all $x \in C - \{0, 1\}$, $f(x, 0) \stackrel{\delta}{=} f(0, x) \stackrel{\delta}{=} g(x, 1) \stackrel{\delta}{=} g(1, x) \stackrel{\delta}{=} x$.*

Proof. Notice that in the proof of Lemma 2.6 we have constructed the binary polynomial h starting from a semilattice operation of $\mathbf{C}|_N/\delta|_N$.

In our case $\mathbf{C}|_N/\delta|_N$ has two semilattice operations, which yields two polynomials f, g with the described properties. $\square$

Definition 2.8. [TCT 2.7] Let $\mathbf{A}$ be an algebra, and B, C be non-void subsets of A . We say that B and C are *polynomially isomorphic* if there are polynomials $f, g \in \text{Pol}_1 \mathbf{A}$ such that $f(B) = C$, $g(C) = B$ and $fg|_C = \text{id}_C$, $gf|_B = \text{id}_B$.

Lemma 2.9. [TCT 4.20] Let $\mathbf{C}$ be a $\langle \delta, \vartheta \rangle$ -minimal algebra, B be the body of $\mathbf{C}$, and N be a $\langle \delta, \vartheta \rangle$ -trace of type **2**. Then $\mathbf{C}$ has a 3-ary polynomial d satisfying:

- (1) $d(x, x, x) = x$, for all $x \in C$.
- (2) $d(b, b, x) = x = d(x, b, b)$, for all $b \in B$ and $x \in C$.
- (3) For every $a, b \in B$, the unary polynomials $d(x, a, b)$, $d(a, x, b)$ and $d(a, b, x)$ are permutations of C .
- (4) B is closed under d .
- (5) Any two $\langle \delta, \vartheta \rangle$ -traces of $\mathbf{C}$ are polynomially isomorphic.

Moreover, every 3-ary polynomial of $\mathbf{C}$ satisfying (1) and (2) also satisfies (3) and (4).

Proof. First we construct a polynomial d that satisfies (1) and (2). Throughout this proof let k be the integer asserted by Lemma 1.4 for C .

CLAIM 1. *There exists $p \in \text{Pol}_3 \mathbf{C}$ such that for all $a \in N$ the functions $p(x, a, a)$, $p(a, a, x)$ are permutations of C , and $p(x, x, x) = x$ for all $x \in C$.*

Since N is of type **2**, there exists $f \in \text{Pol}_3 \mathbf{C}$ so that N is closed under f , and $f(x/\delta, y/\delta, z/\delta) = x/\delta - y/\delta + z/\delta$, that is, f/δ is the Mal'cev operation $x - y + z$ of the vectorspace $\mathbf{C}|_N/\delta|_N$. Clearly, $f(x, a, a) \equiv f(a, a, x) \equiv f(x, x, x) \equiv x \pmod{\delta}$ for all $a, x \in N$. So none of these functions collapse ϑ into δ , therefore they are permutations of C . Put $g(x) = f(x, x, x)$. By Lemma 1.4, $g^k(x) = x$ for all $x \in C$. Finally, define $p(x, y, z) = g^{k-1}f(x, y, z)$. Clearly, $p(x, x, x) = x$. Finally, $p(a, a, x)$ and $p(x, a, a)$ are permutations, because g^{k-1} is a permutation of C .

CLAIM 2. *For all $b \in B$, the functions $p(x, b, b)$ and $p(b, b, x)$ are permutations of C .*

If $b \in N$ then the assertion is true by the previous claim. Now assume that $b \notin N$, and fix an element $a \in N$. Then $p(x, a, a)$ is

a permutation of C , therefore $p_{(0)}^k(x, a, a) = x$. Take an element c such that $\langle c, b \rangle \in \vartheta - \delta$. Clearly, $p_{(0)}^k(c, a, a) = c$. On the other hand $p_{(0)}^k(c, c, c) = c$, as $p(c, c, c) = c$. This shows that the function $p_{(0)}^k(c, x, x)$ is not a permutation, therefore it collapses ϑ into δ . In particular, $p_{(0)}^k(c, b, b) \equiv p_{(0)}^k(c, c, c) = c \not\equiv b = p_{(0)}^k(b, b, b) \pmod{\delta}$. Therefore the function $p_{(0)}^k(x, b, b)$ is a permutation of C , hence so is $p(x, b, b)$. A similar argument works for the other function.

CLAIM 3. *There exists $q \in \text{Pol}_3 \mathbf{C}$ such that $q(x, x, x) = x = q(x, b, b)$ for all $x \in C$, and $q(b, b, x)$ is a permutation of C , for all $b \in B$.*

Since the function $p(x, b, b)$ is a permutation of C , $p_{(0)}^k(x, b, b) = x$ for all $b \in B$. Define $q(x, y, z) = p_{(0)}^{k-1}(p(x, y, z), y, y)$. Clearly, $q(x, x, x) = x = q(x, b, b)$ for all $b \in B$. Finally the function $q(b, b, x)$ is a permutation of C because both $p(b, b, x)$ and $p_{(0)}^{k-1}(x, b)$ are permutations.

CLAIM 4. *There exists $d \in \text{Pol}_3 \mathbf{C}$ satisfying (1) and (2).*

Define $d(x, y, z) = q_{(2)}^{k-1}(x, x, q(x, y, z))$. For all $b \in B$, the function $q(b, b, x)$ is a permutation of C , therefore $d(b, b, x) = x$. Clearly, $d(x, x, x) = x$. Finally, $d(x, b, b) = q_{(2)}^{k-1}(x, x, q(x, b, b)) = q_{(2)}^{k-1}(x, x, x) = x$, for all $b \in B$.

In the rest of the proof let d be any 3-ary polynomial satisfying (1) and (2). We shall prove (3) and (4) for this d .

CLAIM 5. *For every $a, b \in B$, the functions $d(a, b, x)$, $d(a, x, b)$ and $d(x, a, b)$ are permutations of C .*

Let us fix the elements $a, b \in B$. First assume that $N = C$, that is, $\mathbf{C}/\delta$ is a vector space over a field F . Then $(d/\delta)(x_1, x_2, x_3) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4$ for some constants $\alpha_i \in F$. Since d satisfies (2), d/δ must depend on all of its three variables. Therefore $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$. This shows that none of the three functions in the claim can collapse ϑ into δ , hence they are all permutations.

Now assume that $N \neq C$. Pick an element c such that $\langle c, b \rangle \in \vartheta - \delta$. Then $d(a, b, x) \in \text{Sym } C$ iff $d(a, b, c) \not\equiv d(a, b, b) = a = d(a, c, c) \pmod{\delta}$ iff $d(a, x, c) \in \text{Sym } C$. Clearly, $d(a, x, c) \equiv d(a, x, b) \pmod{\vartheta}$ for all $x \in C$. Since ϑ has at least two distinct classes, $d(a, x, c) \in \text{Sym } C$ iff $d(a, x, b) \in \text{Sym } C$. Hence we can conclude that $d(a, b, x) \in \text{Sym } C$ iff $d(a, x, b) \in \text{Sym } C$. A dual argument yields that this happens if and only if $d(x, a, b) \in \text{Sym } C$.

Suppose that they all fail to be permutations. Consider the polynomial $h(x) = d(a, d(a, x, b), x)$. Clearly, $h(b) = b$. On the other hand $h(c) = d(a, d(a, c, b), c) \equiv d(a, a, c) = c \pmod{\delta}$ because $d(a, c, b) \equiv d(a, b, b) = a \pmod{\delta}$. This shows that $h \in \text{Sym } C$. If $x \in N$, then $d(a, x, b) \equiv d(a, a, b) = b \pmod{\delta}$, and $h(x) \equiv d(a, b, x) \equiv d(a, b, a) \pmod{\delta}$. This shows that h collapses $N \times N$ into δ , which is a contradiction to that $h \in \text{Sym } C$.

CLAIM 6. *B is closed under d .*

Take elements $a, b, c \in B$. Then the polynomial $d(a, b, x)$ is a permutation of C , therefore for each congruence γ of $\mathbf{C}$, it maps γ -classes onto γ -classes. In particular, it must map $\langle \delta, \vartheta \rangle$ -traces onto $\langle \delta, \vartheta \rangle$ -traces, hence $d(a, b, c) \in B$.

CLAIM 7. *Any two $\langle \delta, \vartheta \rangle$ -traces N and K are polynomially isomorphic.*

Take $a \in N$ and $b \in K$. Then the polynomial $s(x) = d(b, a, x)$ maps a to b , hence N onto K . Clearly, its inverse s^{k-1} is also a polynomial. $\square$

Corollary 2.10. *Let $\mathbf{C}$ be a $\langle \delta, \vartheta \rangle$ -minimal algebra. Then all $\langle \delta, \vartheta \rangle$ -traces of $\mathbf{C}$ have the same type.* $\square$

Definition 2.11. [TCT 4.21] Let $\mathbf{C}$ be a $\langle \delta, \vartheta \rangle$ -minimal algebra. We say that $\mathbf{C}$ is of type **1, 2, 3, 4** or **5** relative to $\langle \delta, \vartheta \rangle$ iff each (or any) $\langle \delta, \vartheta \rangle$ -trace of $\mathbf{C}$ is of type **1, 2, 3, 4** or **5**, respectively.

Definition 2.12. [TCT 4.16, 4.18, 4.22] If $\mathbf{C}$ is a $\langle \delta, \vartheta \rangle$ -minimal algebra of type **5**, then any binary polynomial satisfying the statement of Lemma 2.6 will be called *pseudo-meet operation of $\mathbf{C}$ with respect to $\langle \delta, \vartheta \rangle$* . Similarly, if $\mathbf{C}$ is of type **3** or **4**, then operations satisfying the statement of Corollary 2.7 are called *pseudo-meet* and *pseudo-join operations*. If $\mathbf{C}$ is of type **2**, then any 3-ary polynomial d satisfying the statement of Lemma 2.9 will be called *pseudo-Mal'cev operation of $\mathbf{C}$ with respect to $\langle \delta, \vartheta \rangle$* .

One interesting consequence of Lemma 2.6 is that if $\mathbf{C}$ is $\langle \delta, \vartheta \rangle$ -minimal and of type **3, 4** or **5**, then δ must be covered by ϑ . This is not necessarily true for types **1** and **2**.

We know that if $\mathbf{C}$ is of type **2, 3, 4** or **5** then all $\langle \delta, \vartheta \rangle$ -traces are polynomially isomorphic. This does not necessarily hold for type **1**. However it is true when δ is covered by ϑ .

Lemma 2.13. *Let $\mathbf{C}$ be a $\langle \delta, \vartheta \rangle$ -minimal algebra, and $\delta \prec \vartheta$ be a congruence cover. Then the $\langle \delta, \vartheta \rangle$ -traces are pairwise polynomially isomorphic.*

Proof. Let N and K be $\langle \delta, \vartheta \rangle$ -traces, and take $\langle a, b \rangle \in \vartheta|_N - \delta$. Since ϑ covers δ , $\delta \vee \text{Cg}_{\mathbf{C}}(\langle a, b \rangle) = \vartheta$. Since K is a ϑ -class, it is contained in $\delta \vee \text{Cg}_{\mathbf{C}}(\langle a, b \rangle)$. Therefore, there exists a polynomial $p \in \text{Pol}_1 \mathbf{C}$ such that $\langle p(a), p(b) \rangle \in \vartheta|_K - \delta$. Since $\mathbf{C}$ is $\langle \delta, \vartheta \rangle$ -minimal, p is a permutation on C . Each permutation maps traces onto traces, hence $p(N) = K$.

Similarly, there exists a permutation $q \in \text{Pol}_1 \mathbf{C}$ such that $q(K) = N$. Now qp is a permutation on C , hence $(qp)^k = \text{id}$ for some $k > 0$. The polynomials $p(qp)^{k-1}$ and q have the required properties. $\square$

Lemma 2.14. [TCT 2.4] *Let $\mathbf{C}$ be a $\langle \delta, \vartheta \rangle$ -minimal algebra, and N be a $\langle \delta, \vartheta \rangle$ -trace of $\mathbf{C}$. Then the map $\alpha \mapsto (\alpha|_N)/\delta|_N$ is a lattice homomorphism of the interval $[\delta, \vartheta]$ onto the congruence lattice of the minimal algebra $\mathbf{C}|_N/\delta|_N$.*

Proof. We will prove more, that the restriction map $|_N : \alpha \mapsto \alpha|_N$ is a lattice homomorphism of $[0_{\mathbf{C}}, \vartheta]$ onto $\text{Con } \mathbf{C}|_N$. This clearly implies the assertion of the lemma. Recall that an equivalence relation is a congruence iff it is closed under all unary polynomials. It is obvious that $\alpha|_N \in \text{Con } \mathbf{C}|_N$ whenever $\alpha \in \text{Con } \mathbf{C}$, and that $|_N$ preserves meets of congruences. Since N is a ϑ -block, $|_N$ also preserves joins of pairs of elements in $[0_{\mathbf{C}}, \vartheta]$. To see that $|_N$ is onto, take $\beta \in \text{Con } \mathbf{C}|_N$, and put

$$\hat{\beta} = \{ \langle x, y \rangle \in \vartheta : \text{for all } p \in \text{Pol}_1 \mathbf{C} \\ \text{if } \{p(x), p(y)\} \subseteq N \text{ then } \langle p(x), p(y) \rangle \in \beta \}.$$

It is easy to see that $\hat{\beta} \in \text{Con } \mathbf{C}$, $\hat{\beta} \leq \vartheta$, and that $\hat{\beta}|_N \leq \beta$. Now take a pair $\langle x, y \rangle \in \beta$. Clearly, $\langle x, y \rangle \in \vartheta$, since $\beta \subseteq N^2 \subseteq \vartheta$. If $p(x) \in N$ for some $p \in \text{Pol}_1 \mathbf{C}$, then $p(N) \subseteq N$ and $p|_N \in \text{Pol}_1 \mathbf{C}|_N$. Therefore $\langle p(x), p(y) \rangle \in \beta$. This shows that $\langle x, y \rangle \in \hat{\beta}|_N$, hence $\hat{\beta}|_N = \beta$. $\square$

Lemma 2.15. *Let $\mathbf{C}$ be a $\langle \delta, \vartheta \rangle$ -minimal algebra, N be a $\langle \delta, \vartheta \rangle$ -trace. If the $\langle \delta, \vartheta \rangle$ -traces are pairwise polynomially isomorphic, then the interval $[\delta, \vartheta]$ of $\text{Con } \mathbf{C}$ is isomorphic to the congruence lattice of $\mathbf{C}|_N/\delta|_N$.*

Proof. Consider the map $\alpha \mapsto (\alpha|_N)/\delta|_N$. By the previous lemma it is enough to show that it is injective. Take congruences $\delta \leq \alpha < \beta \leq \vartheta$, and a pair $\langle x, y \rangle \in \beta - \alpha$. Clearly, $\langle x, y \rangle \in \vartheta - \delta$. Let K be the $\langle \delta, \vartheta \rangle$ -trace containing x and y . Since K is polynomially isomorphic to N , we have a permutation $q \in \text{Pol}_1 \mathbf{C}$ so that $q(K) = N$ and $\langle q(x), q(y) \rangle \in$

$\vartheta - \delta$. All congruences of $\mathbf{C}$ are invariant under q , hence $\langle q(x), q(y) \rangle \in \beta|_N - \alpha|_N$. This proves that $(\alpha|_N)/\delta|_N < (\beta|_N)/\delta|_N$. $\square$

3. TAME QUOTIENTS AND $\langle \delta, \vartheta \rangle$ -MINIMAL SETS

Definition 3.1. [TCT 2.5] Let $\langle \delta, \vartheta \rangle$ be a congruence quotient in a finite algebra $\mathbf{A}$. We define $M_{\mathbf{A}}(\delta, \vartheta)$ to be the minimal members of

$$\{p(A) : p \in \text{Pol}_1 \mathbf{A} \text{ and } p(\vartheta) \not\subseteq \delta\}.$$

The members of $M_{\mathbf{A}}(\delta, \vartheta)$ are called $\langle \delta, \vartheta \rangle$ -minimal sets of $\mathbf{A}$.

Observe that the set $M_{\mathbf{A}}(\delta, \vartheta)$ is always non-empty, and for each $\langle \delta, \vartheta \rangle$ -minimal set U , we have $\vartheta|_U \not\subseteq \delta|_U$.

Definition 3.2. A congruence quotient $\langle \delta, \vartheta \rangle$ is *tame*, if there exists a minimal set $U \in M_{\mathbf{A}}(\delta, \vartheta)$ such that restriction map $|_U : \alpha \mapsto \alpha|_U$ is a 0, 1-separating lattice homomorphism of the interval $[\delta, \vartheta]$ onto $[\delta|_U, \vartheta|_U] \subseteq \text{Con } \mathbf{A}|_U$.

Lemma 3.3. Let $\langle \delta, \vartheta \rangle$ be a tame congruence quotient of a finite algebra $\mathbf{A}$. Then for all $V \in M_{\mathbf{A}}(\delta, \vartheta)$ the following hold.

- (1) $\mathbf{A}|_V$ is a $\langle \delta|_V, \vartheta|_V \rangle$ -minimal algebra.
- (2) There exists $e \in E(\mathbf{A})$ such that $V = e(A)$.
- (3) $\delta \vee \text{Cg}_{\mathbf{A}}(\vartheta|_V) = \vartheta$.
- (4) The restriction map $|_V : \alpha \mapsto \alpha|_V$ is a 0, 1-separating lattice homomorphism of the interval $[\delta, \vartheta]$ onto $[\delta|_V, \vartheta|_V] \subseteq \text{Con } \mathbf{A}|_V$.
- (5) The $\langle \delta, \vartheta \rangle$ -minimal sets of $\mathbf{A}$ are pairwise polynomially isomorphic.

4. ABELIAN ALGEBRAS

Definition 4.1. [TCT 3.3] Let α, β, γ be congruences of an algebra $\mathbf{A}$. We say that α *centralizes* β *modulo* γ , and use the formula $C(\alpha, \beta; \gamma)$, if for every $n \geq 1$, for every $p \in \text{Pol}_{n+1} \mathbf{A}$, and for all pairs $\langle u, v \rangle \in \alpha$, and $\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle \in \beta$ the following equivalence holds:

$$\begin{aligned} p(u, a_1, \dots, a_n) &\stackrel{\gamma}{\equiv} p(u, b_1, \dots, b_n) \\ &\Updownarrow \\ p(v, a_1, \dots, a_n) &\stackrel{\gamma}{\equiv} p(v, b_1, \dots, b_n). \end{aligned}$$

Definition 4.2. [TCT 3.6] Let $\langle \alpha, \beta \rangle$ be a congruence quotient in an algebra $\mathbf{A}$. We say that β is *Abelian over* α in $\mathbf{A}$ if $C(\beta, \beta; \alpha)$, and that $\mathbf{A}$ is *Abelian* if $C(1_{\mathbf{A}}, 1_{\mathbf{A}}, 0_{\mathbf{A}})$.

It is not hard to see that β is Abelian over α in $\mathbf{A}$ if and only if β/α is Abelian over $0_{\mathbf{A}/\alpha}$ in $\mathbf{A}/\alpha$.

Lemma 4.3. [TCT 4.13, 4.14] *Let $\mathbf{C}$ be $\langle \delta, \vartheta \rangle$ -minimal. Then ϑ is Abelian over δ in $\mathbf{C}$ iff the minimal algebras $\mathbf{C}|_N/\delta|_N$ are Abelian for all $\langle \delta, \vartheta \rangle$ -traces N of $\mathbf{C}$.*

Proof. One direction is trivial. Conversely, assume that $C(\vartheta, \vartheta; \delta)$ fails in $\mathbf{C}$ for some polynomial $p \in \text{Pol}_{n+1} \mathbf{C}$ and elements $u \equiv v, a_1 \equiv b_1, \dots, a_n \equiv b_n \pmod{\vartheta}$. We can assume that $\delta = 0_{\mathbf{C}}$, since by taking the quotient by δ we do not change Abelian-ness and the set of minimal algebras that correspond to traces. So we have the following failure: $p(u, \bar{a}) = p(u, \bar{b})$ and $p(v, \bar{a}) \neq p(v, \bar{b})$.

We can also assume that p is of minimal arity. Clearly, $u \neq v$, and by the minimality of p , $a_i \neq b_i$ for each i . Thus the classes $N_0 = v/\vartheta$, $N_i = a_i/\vartheta$ ($1 \leq i \leq n$), and $K = p(v, \bar{a})/\vartheta$ are $\langle 0, \vartheta \rangle$ -traces of $\mathbf{C}$. It follows that $p(N_0, N_1, \dots, N_n) \subseteq K$.

Since $p(u, \bar{a}) \neq p(v, \bar{a})$ or $p(u, \bar{b}) \neq p(v, \bar{b})$, the mapping $p : N_0 \times N_1 \times \dots \times N_n \rightarrow K$ depends on its first variable. Since p is of minimal arity, this mapping depends on its other variables. Thus for each $0 \leq i \leq n$ there exists $\bar{c} \in N_0 \times N_1 \times \dots \times N_n$ such that $\alpha_i(x) = p(c_0, c_1, \dots, c_{i-1}, x, c_{i+1}, \dots, c_n)$ is non-constant and $\alpha_i(N_i) \subseteq K$. Therefore α_i is a permutation which maps N_i onto K . Now $\alpha_i^{-1} \in \text{Pol}_1 \mathbf{C}$ and $\alpha_i^{-1}(K) = N_i$.

Put $f(x_0, x_1, \dots, x_n) = p(\alpha_0^{-1}(x_0), \dots, \alpha_n^{-1}(x_n))$. Now $f(K^{n+1}) = K$, and $f|_K$ exhibits the failure of $C(1_{\mathbf{C}|_N}, 1_{\mathbf{C}|_N}; 0_{\mathbf{C}|_N})$ in $\mathbf{C}|_N$ with the elements $\alpha_0(u), \alpha_0(v)$ and $\alpha_i(a_i), \alpha_i(b_i)$ ($1 \leq i \leq n$). $\square$

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