Generalized Effect Algebras as Models of Concurrent Resources

Peter Jipsen   M. Andrew Moshier

School of Computational Sciences and
Center of Excellence in Computation, Algebra and Topology (CECAT)
Chapman University, Orange, California

BLAST2015@UNT

Outline

- Small partial groupoids
- Separation algebras
- Applications

Introduction

A **groupoid** is a set A with a binary operation $\cdot : A \times A \to A$

If A is finite, say $= \{a_1, \ldots, a_n\}$, then a groupoid can be defined by its operation table

$\cdot$	a_1	a_2	$\cdots$	a_n
a_1				
a_2				
$\vdots$				
a_n				

$\leftarrow$ fill out with (some of) $a_1, \ldots, a_n$ any way you like

Introduction

- What do we know about 2-element groupoids?
- How many are there? 16 Up to isomorphism? 10
-

$\cdot$	0	1
0		
1		

C	0	1
0	0	0
1	0	0

$\cdot$	0	1
0	0	0
1	0	1

$\leftarrow$	0	1
0	0	0
1	1	0

π_1	0	1
0	0	0
1	1	1

$\rightarrow$	0
0	0
1	0

- $xy = zw$ | assoc. comm. idem. | implic. reduct of BA | $xy = x$ (Lz sg) | implic. reduct of BA

-

π_2	0	1
0	0	1
1	0	1

$+$	0	1
0	0	1
1	1	0

$\mid$	0	1
0	1	0
1	0	0

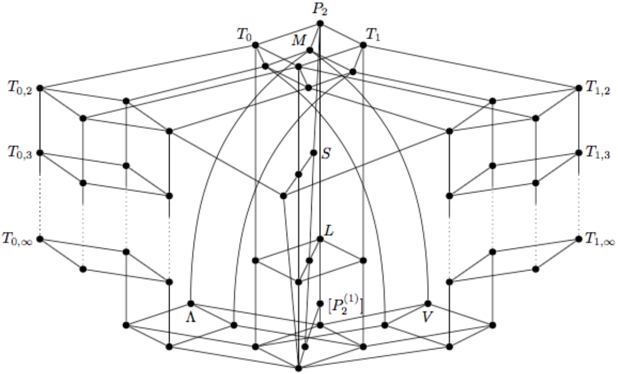
$\bar{\pi}_2$	0	1
0	1	0
1	1	0

$\bar{\pi}_1$	0	1
0	1	1
1	0	0

- $xy = y$ (Rz sg) | assoc. $x + x = 0$ $x + 0 = x$ | BA axioms | $xy = yy$ $x^2x^2 = x$ | $xy = xx$ $x^2x^2 = x$
- Give axioms for the varieties they generate

Introduction

In fact, quite a bit is known about 2-element algebras



Post lattice (from Schölzel 2010)

Introduction

A **partial groupoid** is a set A with a **partial** binary operation $\cdot : A \times A \rightsquigarrow A$

If $A = \{a_1, \ldots, a_n\}$, then a partial groupoid can be defined by a **partially filled out** operation table

$\cdot$	a_1	a_2	$\cdots$	a_n
a_1				
a_2				
$\vdots$				
a_n				

What do we know about 2-element partial groupoids?

- How many are there? $81 - 16 = 65$ Up to isomorphism? $45 - 10 = 35$

-

$\cdot_1$	0	1
0	0	0
1	0	

$\cdot_2$	0	1
0	0	0
1	1	

$\cdot_3$	0	1
0	0	0
1		0

$\cdot_4$	0	1
0	0	0
1		1

$\cdot_5$	0	1
0	0	0
1		

$\cdot_6$	0	1
0	0	1
1	0	

$\cdot_7$	0	1
0	0	1
1	1	

$\cdot_8$	0	1
0	0	1
1		0

$\cdot_9$	0	1
0	0	1
1		1

$\cdot_{10}$	0	1
0	0	1
1		

$\cdot_{11}$	0	1
0	0	
1	0	0

$\cdot_{12}$	0	1
0	0	
1	0	

$\cdot_{13}$	0	1
0	0	
1	1	0

$\cdot_{14}$	0	1
0	0	
1	1	

$\cdot_{15}$	0	1
0	0	
1		0

$\cdot_{16}$	0	1
0	0	
1		1

$\cdot_{17}$	0	1
0	0	
1		

$\cdot_{18}$	0	1
0	1	0
1	0	

$\cdot_{19}$	0	1
0	1	0
1	1	

$\cdot_{20}$	0	1
0	1	0
1		0

$\cdot_{21}$	0	1
0	1	0
1		

$\cdot_{22}$	0	1
0	1	1
1	0	

$\cdot_{23}$	0	1
0	1	1
1	1	

$\cdot_{24}$	0	1
0	1	1
1		0

$\cdot_{25}$	0	1
0	1	1
1		

$\cdot_{26}$	0	1
0	1	
1	0	

$\cdot_{27}$	0	1
0	1	
1	1	

$\cdot_{28}$	0	1
0	1	
1		0

$\cdot_{29}$	0	1
0	1	
1		

$\cdot_{30}$	0	1
0		0
1	0	

$\cdot_{31}$	0	1
0		0
1	1	

$\cdot_{32}$	0	1
0		0
1		

$\cdot_{33}$	0	1
0		1
1	0	

$\cdot_{34}$	0	1
0		1
1		

$\cdot_{35}$	0	1
0		
1		

Axioms for the ISP classes they generate?

- Which of these generate a **finitely axiomatizable** ISP class?
- **Natural duality theory** applies to partial algebras (*Davey* [2006])
- Which of these partial groupoids are **dualizable**?
- For this talk: want the operation to have an **identity element** 0
- Two **total** groupoids: 2-element semilattice and 2-element group
-

$\vee$	0	1
0	0	1
1	1	1

$+_2$	0	1
0	0	1
1	1	0

- How many **partial groupoids with an identity element** are there?
- $P_1 =$

$+$	0	1
0	0	1
1	1	

Is ISP(P_1) **finitely axiomatizable**? **Dualizable**?

Subalgebras, products, homomorpisms

- A **partial groupoid** is a set P with a partial binary operation $*: P \times P \rightsquigarrow P$
- Ljapin and Evseev [1997] call it a **pargoid** for short
- Every pargoid can be extended to a groupoid $\tilde{P}$ by adding one element $\infty \notin P$
- $x \tilde{*} y = x * y$ if $x * y$ is defined, and $x \tilde{*} y = \infty$ otherwise
- $\mathrm{dom}(*) = \{(x, y) \mid x * y \neq \infty\}$
- $Q \subseteq P$ is a **(partial) subalgebra** if Q is closed under existing products
- $\prod_{i \in I} Q_i$ is the **product**, $*$ defined pointwise, exits **iff** it exists in all coords
- Note that $\tilde{P} \times \tilde{Q} \neq \widetilde{P \times Q}$
- $h : P \to Q$ is a **homomorphism** if $h(x * y) = h(x) * h(y)$ for all $(x, y) \in \mathrm{dom}(*)$
- **HSP** is defined using these operations

Identities and quasiidentities

Terms and formulas are defined as usual

A term $t(x_1, \ldots, x_n)$ is defined iff all subterms are defined

An identity $s(x_1, \ldots, x_n) = t(x_1, \ldots, x_n)$ holds in a partial algebra P if for all $x_1, \ldots, x_n \in P$ either both sides are undefined, or they are defined and equal

A quasiidentity $s_1 = t_1 \& \cdots \& s_n = t_n \implies s = t$ holds in P if for all assignments that make the premises defined and equal, s, t are defined and equal

Why bother with partial operations?

- Boole originally considered $\cup$ undefined for overlapping sets
- Products of partial algebras are cartesian (not true with $\tilde{P}$)
- Natural duality **now allows partial operations** and relations on both sides
- The main reason: **Computer Science**
- Consider the memory of a computer: a list of **cells** with **values** in them
- $(m_0, m_1, \ldots, m_i, \ldots)$ or more generally:
- a function $m : L \to V$ from a set L of **locations** to V of **values**
- As a program runs, it is allocated some of these cells
- The part of memory used is called a **heap** h, where $h : L \rightsquigarrow V$ is a **partial** function
- If several programs run **concurrently**, they use **separate** heaps

Separation Algebras

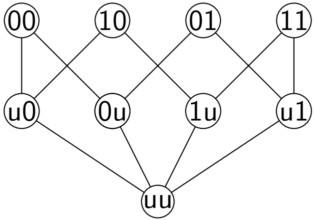
Separation algebras

- *Calcagno, O'Hearn, Yang* [2007] define a **separation algebra** to be a cancellative commutative **partial** monoid
- i.e., of the form $(A, +, 0)$ where $+ : A \times A \rightsquigarrow A$ is
- $x + y = y + x$ (commutative)
- $(x + y) + z = x + (y + z)$ (associative)
- $x + 0 = x$ (0 is the identity)
- $x + z = y + z \implies x = y$ (cancellative)
- Typical model: $P_{L,V} = \{$ all heaps ($=$ partial functions L to V) $\}$ and $h + k = \begin{cases} h \cup k & \text{if } \mathrm{dom}(h) \cap \mathrm{dom}(k) = \emptyset \\ \text{undef.} & \text{otherwise} \end{cases}$
- E.g., if $L = \{0, 1\}$ and $V = \{0, 1\}$ we have $A = \{uu, u0, u1, 0u, 1u, 00, 01, 10, 11\}$ where $h = ab$ is the heap that satisfies $h(0) = a$, $h(1) = b$; $h(x) = u$ means undefined
- Define **heap algebras** $= \mathsf{S}(\{P_{L,V} \mid L, V \text{ are sets}\})$. Note that $0 = uu$

Heap algebra examples

- $P_{2,2} =$

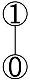
$+$	uu	$u0$	$u1$	$0u$	$1u$	00	01	10	11
uu	uu	$u0$	$u1$	$0u$	$1u$	00	01	10	11
$u0$	$u0$			00	10				
$u1$	$u1$			01	11				
$0u$	$0u$	00	01						
$1u$	$1u$	10	11						
00	00								
01	01								
10	10								
11	11								



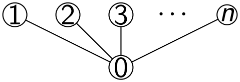
- Define $x \leq y$ if $x + z = y$ for some z (the natural order)
- Can you find another (smaller) example? Guess what! $P_1 = P_{1,1}$
- $P_1 =$

$+$	0	1
0	0	1
1	1	

$=$



$P_{1,n} =$



Heap algebras = ISP(P_1)

Products of P_1 are Boolean lattice reducts with $x+y=x\vee y$ if $x\wedge y=0$

What do the (partial) subalgebras of products of P_1 look like?

Theorem

The class of heap algebras is ISP(P_1)

Proof.

$P_{L,V}\cong(V\cup\{u\})^L$ where $u\notin V$

1. $P_{1,V}$ is a subalgebra of $(P_1)^V$
2. Observe that $P_{L,V}=(P_{1,V})^L$

□

What is known about separation algebras

Let SA = quasivariety of separation algebras (canc. comm. partial monoids)

SA is larger than $\mathsf{ISP}(P_1)$ since $\mathbb{Z}_2 \in \mathsf{SA}$

P_1 is **positive**, i.e., satisfies $x + y = 0 \implies x = 0$, which fails in $\mathbb{Z}_2$

Positive separation algebras are also know as **generalized effect algebras**

Fact

*$x \leq y$ is a **partial order** in positive separation algebras*

Effect algebras come from quantum logic, *Foulis and Bennett* [1994]

Effect algebras are positive SAs that have unary $'$ such that $x + x' = 0'$

Orthogonal separation algebras

Addition in P_1 is **orthogonal**, i.e., $x + x = x + x \implies x = 0$

Unit interval with truncated $+$ is a **non-orthogonal** pos. SA: $\frac{1}{2} + \frac{1}{2} = 1$

Lemma: Orthogonal separation algebras are positive.
Proof: If $x + y = 0$ then $x + (x + y)$ is defined, so $(x + x) + y$ is defined, hence $x + x = x + x$ so we get $x = 0$

Orthogonal separation algebras are also known as **generalized orthoalgebras**

P_1 is **coherent**, i.e., if $x + y$, $x + z$ and $y + z$ are defined, so is $(x + y) + z$

Example of a **non-coherent** orthogonal separation algebra: Take an 8-element BA and remove the top element

Coherent orthoalgebras are f.o. equivalent to orthomodular posets

Concrete separation algebras

Let U be any set and define $+$ on $\mathscr{P}(U)$ by

$$X+Y=\begin{cases}X\cup Y & \text{if } X\cap Y=\emptyset\\ \text{undef.} & \text{otherwise}\end{cases}$$

Then $(\mathscr{P}(U),+,\emptyset)$ is a coherent orthogonal separation algebra $\cong {P_1}^{|U|}$

A **concrete separation algebra** is any partial algebra embedded in this powerset algebra

Hence the class of **concrete separation algebras** is $\mathsf{ISP}(P_1)$

But this is **smaller** than the class of coherent orthogonal separation algebras

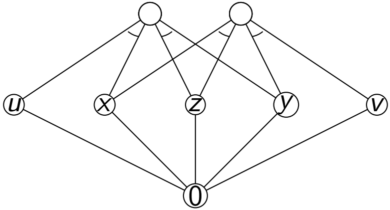
Is it **finitely axiomatizable**?

New quasiidentities

Lemma: There are at least two more quasiequations that hold in P_1 and are not consequences of previous axioms:

1. $x \leq y + z \;\&\; y \leq x + z \implies x = y$
2. $w + x = y + z \;\&\; w + y = u \;\&\; x + z = v \implies x = y$

Proof that 1. holds in P_1: Suppose $x \leq y + z \;\&\; y \leq x + z$ but $x \neq y$.
By symmetry can assume $x = 0$, $y = 1$.
Then $x \leq y + z$ implies $z = 0$ (since $y + z$ must be defined).
But now $1 = y \leq x + z = 0 + 0 = 0$ is a contradiction.
1. fails in this coherent orthogonal SA:



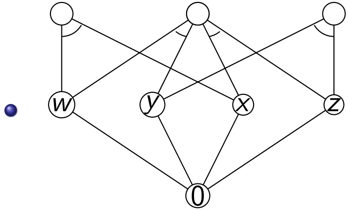
where only $u + x, y + z, x + z, y + v$ are defined

Similarly 2. $w + x = y + z$ & $w + y = u$ & $x + z = v \implies x = y$ holds in P_1: Suppose $x = 0, y = 1$.

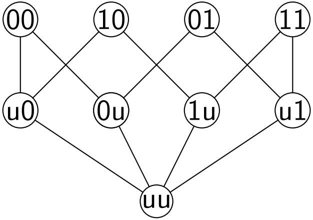
$w + x = y + z$ implies $z = 0$ and $w = 1$.

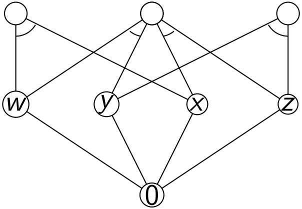
But now $w + y = 1 + 1$ is undefined, contradicting $w + y = u$.

Below is a coherent orthogonal SA that satisfies 1. but fails 2.



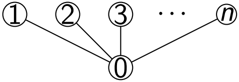
ISP(P_1) is not closed under H

- $(P_{1,2})^2 \cong$ 

- and has a homomorphic image 

$\notin \mathrm{ISP}(P_1)$

Heap algebras satisfy no congruence equations

- Consider the heap algebra $P_{1,n} =$ 

- Can identify any two maximal elements without collapsing any others
- Can identify any maximal element with 0 without collapsing any others
- Therefore $\mathrm{Con}(P_{1,n}) = \mathrm{Eq}(n) =$ the lattice of equivalence relations on an n element set
- Any lattice equation fails in $\mathrm{Eq}(n)$ for some n

Applications

Applications of Separation Algebras

Let L be a set of **memory locations**, and V a set of **values** that they can store.

A **state** of a computation is a partial function $s : L \rightsquigarrow V$

The set S of all states is $(V \cup \{u\})^L$

A **program** P is a binary relation on S, i.e., $P \subseteq S \times S$

The **identity** program $1 = \{(s, s) \mid s \in S\}$; **abort** $= 0 = \{\}$

Operations on programs:

- **Composition** $PQ = P; Q = \{(r, t) \mid \exists s(r, s) \in P \text{ and } (s, t) \in Q\}$
- Nondeterministic **choice** $P + Q = P \cup Q$
- Finite **iteration** $P^* = 1 \cup P \cup PP \cup PPP \cup \cdots$

An abstract algebra of programs

A **Kleene algebra** is of the form $(A, +, 0, \cdot, 1, ^*)$ such that

- $(A, +, 0)$ is a **join semilattice** with bottom,
- $(A, \cdot, 1)$ is a **monoid**,
- $x(y+z) = xy + xz, \qquad (x+y)z = xz + yz,$
- $0x = x0 = 0, \qquad 1 + x + x^*x^* = x^*,$
- $xy \leq y \implies x^*y \leq y$ and $yx \leq y \implies yx^* \leq y$

To handle **if ... then ... else ...** and **while ... do** Kozen [1997] defined

Kleene algebras with test $(A, A', +, 0, \cdot, 1, ^*, \bar{\ })$ where in **addition** $(A', +, 0, \cdot, 1, \bar{\ })$ is a Boolean algebra and $A' \subseteq A$

Now: **if** b **then** p **else** $q = bp + \bar{b}q$ and **while** b **do** $p = (bp)^*\bar{b}$

A computation trace model

Let $S^* = \{s_1 s_2 \cdots s_n \mid s_i \in S \text{ for } i = 1, \ldots, n \in \mathbb{N}\}$, $1 = \{\lambda\}$ where λ is the empty string, and for $u, v \in S^*$

$$\text{define } u \diamond v = \begin{cases} u_1 \cdots u_m v_2 \cdots v_n & \text{if } u_m = v_1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

So for example, $s \diamond s = s$, $sst \diamond tst = sstst$ and $sst \diamond sst =$ undefined

For $P, Q \subseteq S^*$ let $PQ = \{u \diamond v \mid u \in P, v \in Q\}$, $1 = S$, $0 = \emptyset$,

$P + Q = P \cup Q$, $P^* = 1 \cup P \cup PP \cup PPP \cup \cdots$

For $B \subseteq S$, let $\bar{B} = S - B$

Then $(\mathscr{P}(S^*), \mathscr{P}(S), +, 0, \cdot, 1, ^*, \bar{\ })$ is a Kleene algebra **with tests**

Want to add **concurrent composition** to this model

Concurrent Kleene algebra with tests

Recall that the set of states S contains all partial functions from L to V

So $(S, \oplus, 0)$ is the heap algebra $P_{L,V}$

Let $(F_S, \cdot, ||, 1)$ be the **free bi-monoid** with generators S and $x||y = y||x$

Note: $\cdot$ and $|$ have the **same** identity 1, but **no other interaction**

$F_{\{\alpha,\beta\}} = \{1, \alpha, \beta, \alpha\alpha, \alpha\beta, \beta\alpha, \beta\beta, \alpha||\alpha, \alpha||\beta, \beta||\beta, \alpha||\alpha\alpha, \alpha(\alpha||\alpha), \ldots\}$

Define the set of **traces** $G_S = S \cup \{\alpha u\beta \mid \alpha, \beta \in S, u \in F_S\}$

and let $\alpha u\beta|\gamma v\delta = (\alpha \oplus \gamma)(u||v)(\beta \oplus \delta)$ if $\alpha \oplus \gamma$, $\beta \oplus \delta$ defined

$\alpha|\gamma = \alpha \oplus \gamma$ and $\alpha|\gamma v\delta = (\alpha \oplus \gamma)v(\alpha \oplus \delta)$ (if defined)

Concurrent Kleene algebra with tests

Sequential composition of traces is as before:

$\alpha u\beta \diamond \gamma v\delta = \alpha u\beta v\delta$ if $\beta = \gamma$, undefined otherwise

Lift $\diamond$, $|$ to **sets of traces** by $P \cdot Q = \{\alpha u\beta \diamond \gamma v\delta \mid \alpha u\beta \in P, \gamma v\delta \in Q\}$

$P||Q = \{(\alpha u\beta|\gamma v\delta) \mid \alpha u\beta \in P, \gamma v\delta \in Q\}, \quad 1 = S, \quad \bar{B} = S - B$

$(\mathscr{P}(G_S), \mathscr{P}(S), \cup, \emptyset, \cdot, ||, ^*, \bar{\ })$ is a **concurrent Kleene algebra with test**

More general concurrent KATs are defined with an arbitrary **positive SA**

A model of concurrency should also satisfy the **weak exchange law**

$(x||y)(z||w) \leq xz||yw$, which can be done by ordering F_S and S and using only upsets of G_S for the algebra

Bunched implication algebras

A **bunched implication algebra** (BI-algebra) is of the form

$(A, \vee, \wedge, \to, \top, \bot, *, \backslash, /, 1)$ where $(A, \vee, \wedge, \to, \top, \bot)$ is a **Heyting algebra**

(i.e. a bounded distributive lattice with $x \wedge y \leq z$ iff $y \leq x \to z$) and

$(A, \vee, \wedge, *, \backslash, /, 1)$ is a **commutative residuated lattice**

(i.e. a commutative monoid with $x * y \leq z$ iff $y \leq x \backslash z$ iff $x \leq z/y$)

If $(x \to \bot) \to \bot = x$ we get **classical** BI-algebras

CBI-algebras = commutative residuated Boolean monoids
= *crm*-algebras of Jónsson-Tsinakis [1993]

BI-algebras come from **Separation Logic**, a **Hoare programming logic** for reasoning about pointers and concurrent resources

BI-algebras from separation algebras

Let $(P, \oplus, 0)$ be a **positive separation algebra**
Recall the **natural order** $x \leq y$ iff $\exists z\, x \oplus z = y$

$Up(P)$ is the set of upward closed subsets of P
$=$ a completely distributive complete lattice under intersection and union

Hence $(Up(P), \cup, \cap, \to, P, \emptyset)$ is a **Heyting algebra**

Define $X * Y = \{x \oplus y \mid x \in X, y \in Y\}$,

$X \backslash Y = \{z \mid x \oplus z \in Y \text{ for all } x \in X\}$, $\quad X/Y = Y \backslash X$ and $1 = \{0\}$

Then $(Up(P), \cup, \cap, \to, P, \emptyset, *, \backslash, /, 1)$ is a bunched implication algebra

Let $Up(\mathbf{SA}) = \{Up(P) \mid P \in \mathbf{SA}\}$

Problem: Axiomatize the class $\mathrm{HSP}(Up(\mathbf{SA}))$

Natural duality (briefly)

Is $P_1 = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & \end{array}$ dualizable in the sense of Davey [2006]?

A finite (partial) algebra A is **dualizable** if there exists a (topological) structure $\bar{A}$ on the same set as A such that for all $B \in \mathbf{ISP}(A)$

if we construct the **dual** $\bar{B} = Hom(B, A)$ as a **closed substructure** of $\bar{A}^B$

then the **double dual** $Hom(\bar{B}, \bar{A})$ as a (partial) **subalgebra** of $A^{\bar{B}}$ is **isomorphic to** B

The dual structure $\bar{A}$ has **compatible fundamental (partial) function and relations**

E.g. $\leq = \{00, 01, 11\}$ is compatible with $+$ since $00 + 01 = 01$, $00 + 11 = 11$

D-poset + ? as dual of P_1

Let $D_1 = \begin{array}{c|cc} - & 0 & 1 \\ \hline 0 & 0 & \\ 1 & 1 & 0 \end{array}$, the **truncated minus operation** on $\{0,1\}$

It is an example of a **generalized D-poset**

$- = \{000, 101, 110\}$ is compatible with $+$

$R = \{000, 001, 010, 011, 101, 011, 111\}$ is compatible with $+$

$Parity_n = \{v \in (P_1)^n \mid v \text{ has an even } \# \text{ of 1s}\}$

However, do not know if these (or all compatible) relations are sufficient for a duality

References

C. Calcagno, P.W. O'Hearn, H. Yang, Local Action and Abstract Separation Logic, Proceedings of the 22nd Annual IEEE Symposium on Logic in Computer Science, 2007, 366–378

B. Davey, Natural dualities for structures, Acta Univ. M. Belii Math. 13, 2006, 3–28

D.J. Foulis and M.K. Bennett, Effect algebras and unsharp quantum logics, Foundations of Physics 24, 10, 1994, 1331–1352

B. Jónsson and C. Tsinakis. Relation algebras as residuated Boolean algebras. Algebra Universalis, 30, 1993 469–478

D. Kozen, Kleene algebra with tests, Transactions on Programming Languages and Systems, 19, 3, 1997, 427–443

Thank you!