On the enduring impact of Bjarni Jónsson's research in Lattice Theory, Relation Algebras and Boolean Algebras with Operators, Part 1

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BLAST: A conference in memory of Bjarni Jónsson

Outline

- Early work 1947-1951
- Boolean Algebras with Operators 1951-1952
- Representation of lattices 1954-1960
- Sublattices of free lattices 1961-1977
- Congruence distributivity 1963-1967
- Varieties of lattices 1968-1979
- Arithmetic of ordered sets 1980-1982
- The theory of binary relations 1982-1991
- BAOs and Distributive lattices with operators 1992-2004

A very short biography of Bjarni Jónsson

Born on February 15, 1920 in **Draghals, Iceland**

B. Sc. from UC Berkeley in 1943

Ph. D. from UC Berkeley in 1946 under **Alfred Tarski**

1946-1956 **Brown University**

1956-1966 **University of Minnesota**

1966-1993 **Vanderbilt University, first distinguished professor**

1974 invited speaker at **International Congress of Mathematicians**

1993- **Vanderbilt University, professor emeritus**

2012 elected inaugural fellow of the **American Mathematical Society**

List of Bjarni Jónsson's Ph. D. students

	Name	University	PhD	Descend.
1	Allen Clarke	Brown U.	1951	9
2	Daniel Wagner	Brown U.	1951	
3	Edgar Smith, Jr.	Brown U.	1955	
4	Peter Fillmore	U. Minnesota	1962	48
5	George Monk	U. Minnesota	1966	
6	Frederick Galvin	U. Minnesota	1967	1
7	Thomas Whaley	Vanderbilt U.	1968	
8	Dang Xuan Hong	Vanderbilt U.	1970	
9	Robert Appleson	Vanderbilt U.	1975	
10	Henry Rose	Vanderbilt U.	1980	3
11	Jeh Gwon Lee	Vanderbilt U.	1983	
12	Young Yug Kang	Vanderbilt U.	1987	
13	Peter Jipsen	Vanderbilt U.	1992	
14	John Rafter	Vanderbilt U.	1994	

Bjarni currently has 75 PhD descendants

University of California at Berkeley in the 1940s

Sather Gate, around the same time

Let's time-travel back 70 years

It's 1946 at UC Berkeley, Bjarni is 26
and near the end of his Ph. D. studies.

He's working on ***direct decompositions of algebraic systems***.

Birkhoff's book ***Lattice Theory*** (1st ed)
appeared 6 years earlier. Birkhoff's paper
Subdirect unions in UA only 2 years ago.

Eilenberg and Mac Lane's ***General theory of natural equivalences*** is only 1 year old.

Tarski's paper on $\mathrm{HSP}(\mathcal{K})$ = the variety
generated by $\mathcal{K}$ appears that year, 1946

Early work 1947-1951

Bjarni has just moved to Rhode Island and taken his first position at Brown University

1. B. Jónsson, A. Tarski: **Direct decompositions of finite algebraic systems**, Notre Dame Math. Lectures 5, v+64 pp, 1947

Results from Bjarni's thesis about algebras $(A, +, 0, (f_i)_{i\in I})$ where 0 is an identity of $+$ and $f_i(0, \ldots, 0) = 0$.

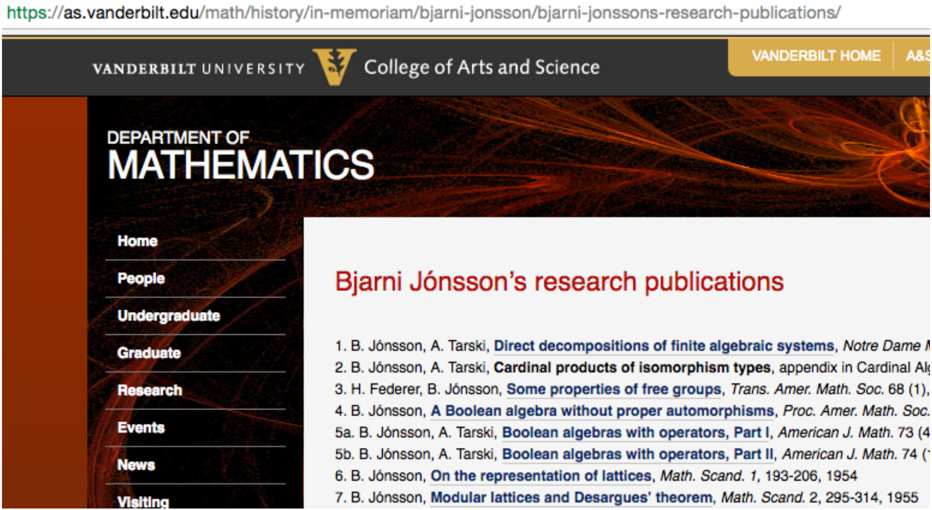
2. B. Jónsson, A. Tarski, **Cardinal products of isomorphism types**, appendix in Cardinal Algebras, Oxford University Press, 253–312, 1949

3. H. Federer, B. Jónsson, **Some properties of free groups**, *Trans. Amer. Math. Soc.* 68 (1), 1–27, 1950

In this talk we only consider a **subset** of Bjarni's research

limited to **some** papers about **lattices**, **posets**, **RAs** and **BAOs**

Bjarni's research publications (PDF files)



This page can be reached from the BLAST 2017 conference homepage

An interesting Boolean algebra

4. B. Jónsson, **A Boolean algebra without proper automorphisms**, *Proc. Amer. Math. Soc.* **2** (5), 766–770, 1951

A finite Boolean algebra is isomorphic to $\mathcal{P}(X)$ for some finite set X

So its automorphism group is the symmetric group S_X

Problem 74 in Birkhoff's **Lattice Theory** (rev. ed. 1949) asked if **every infinite Boolean algebra admits a nontrivial automorphism**.

In his 1951 paper Bjarni constructs a **Boolean space** (= a compact topological space with a basis of clopen sets) that has **no nontrivial homeomorphisms**

So by the Marshall Stone's **categorical duality** between Boolean spaces and Boolean algebras the answer is **no**.

The construction is based on an interval topology of a linear order.

Boolean algebras with operators (BAOs), Part I

5a. B. Jónsson and A. Tarski, **Boolean algebras with operators, Part I**, *American J. Math.* **73** (4), 891–939, 1951

This paper has **824 citations** on Google scholar (37% more than the next)

Modal algebras and relation algebras had been studied for over a decade

BAOs are a generalization of them

They are also a central notion of classical algebraic logic

Contains many deep results that were rediscovered later in modal logic

$\mathbf{A} = (\mathbf{A}_0, (f_i)_{i\in I})$ is a **BAO** if $\mathbf{A}_0 = (A, +, \cdot, ^-, 0, 1)$ is a **Boolean alg.** and

each f_i is an **operator**: $f_i(\ldots, x+y, \ldots) = f_i(\ldots, x, \ldots) + f_i(\ldots, y, \ldots)$

An operator is **normal** if $f(\ldots, x_{j-1}, 0, x_{j+1}, \ldots) = 0$.

Main results of BAO Part I

$\mathbf{A} = (\mathbf{A}_0, f_0, \ldots, f_k \ldots)$ is a **perfect extension** of $\mathbf{B} = (\mathbf{B}_0, g_0, \ldots, g_k \ldots)$ and $\mathbf{B}$ is a **regular subalgebra** of $\mathbf{A}$ if

- $\mathbf{A}$ is **complete** and **atomic**, with subalgebra $\mathbf{B}$
- $1 = \sum_{i \in I} x_i$ implies $1 = \sum_{i \in J} x_i$ for some **finite** subset J of I
- $u \neq v \in \mathrm{At}(\mathbf{A})$ implies $u \leq b$ and $v \cdot b = 0$ for some $b \in \mathbf{B}$
- $f_k(x) = \prod\{g_k(y) : x \leq y \in B^{m_k}\}$ whenever $x \in \mathrm{At}(\mathbf{A}^{m_k})$

In modern terminology, $f_k = g_k^\sigma$ and $\mathbf{A}$ is the **canonical extension** $\mathbf{B}^\sigma$.

Theorem (1951, Thm 2.15, 2.16)

*For any BAO $\mathbf{B}$ there exists a complete and atomic BAO $\mathbf{A}$ (unique up to isomorphism) which is a **perfect extension** of $\mathbf{B}$.*

If $\mathcal{V}$ is a variety defined by $^-$-free identities and $\mathbf{B} \in \mathcal{V}$ then $\mathbf{A} \in \mathcal{V}$.

Main results of BAO Part I

For a relational structure $(U, R_0, \ldots, R_k, \ldots)$ the **complex algebra** is $(\mathcal{P}(U), \cup, \emptyset, \cap, U, R_0^*, \ldots, R_k^*, \ldots)$ where for $X_i \subseteq U$ for $i = 1, \ldots, m_k$

$$R_k^*(X_1, \ldots, X_{m_k}) = \{y \in U : R(x_1, \ldots, x_{m_k}, y) \text{ and } x_i \in X_i\}$$

Theorem (1951, Thm 3.9, 3.10)

*The **complex algebra** of any **relational structure** is a **normal complete** and **atomic BAO**.*

*Conversely every **normal complete** and **atomic BAO** is isomorphic to the **complex algebra** of some **relational structure**.*

*Every **normal BAO** is isomorphic to a **regular subalgebra** of the **complex algebra** of some **relational structure**.*

This result was applied to **closure algebras** (= modal logic S4), **cylindric algebras** and **relation algebras**.

Relation algebras

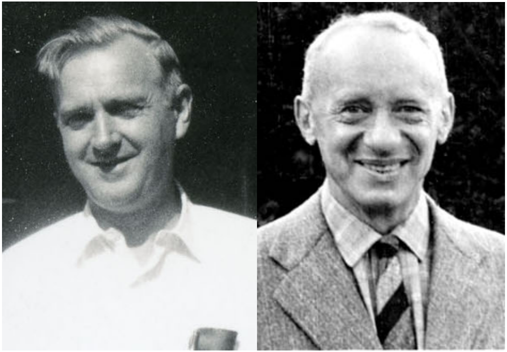
The **calculus of binary relations** was developed by

A. De Morgan [1864], **C. S. Peirce** [1883], and **E. Schröder** [1895]

At the time it was considered one of the cornerstones of mathematical logic

Alfred Tarski [1941] gave a set of axioms, refined in 1943 to 10 equational axioms, for (abstract) relation algebras $\mathbf{A} = (A, +, 0, \cdot, 1, ^-, ;, 1', ^\smile)$

(R1) $x + y = y + x$
(R2) $x + (y + z) = (x + y) + z$
(R3) $\overline{\overline{x} + y} + \overline{\overline{x} + \overline{y}} = x$
(R4) $x; (y; z) = (x; y); z$
(R5) $x; 1' = x$
(R6) $x^{\smile\smile} = x$
(R7) $(x; y)^\smile = y^\smile; x^\smile$
(R8) $(x + y); z = x; z + y; z$
(R9) $(x + y)^\smile = x^\smile + y^\smile$
(R10) $x^\smile; \overline{x; y} + \overline{y} = \overline{y}$



5b. BAOs, Part II, *American J. Math.* **74** (1), 127–162, 1952

A ***relation algebra*** **A** is a Boolean algebra $(A, +, 0, \cdot, 1)$ with an associative binary operation ;, unary $^{\smile}$, and constant $1'$ such that $1'; x = x = x; 1'$ and

$$(x; y) \cdot z = 0 \iff (x^{\smile}; z) \cdot y = 0 \iff (z; y^{\smile}) \cdot x = 0$$

Results about **conjugated operations** imply that every **relation algebra** is a BAO, and that this definition is **equivalent** to (R1)-(R10).

BAOs, Part II: Applications to relation algebras

An algebra is **simple** if it has only two congruences.

Theorem (Thm 4.10,4.14)

A relation algebra $\mathbf{A}$ *is* ***simple*** $\iff$ $\mathbf{A}$ *is* ***directly indecomposable***

$\iff$ $x \neq 0$ *implies* $1;x;1 = 1$ *for every* $x \in A$.

An element x in a relation algebra is **functional** if $x^{\smile};x \leq 1'$.

Theorem (Thm 5.6)

Let $\mathbf{A}$ *be an* ***atomic relation algebra*** *with* $0 \neq 1$ *in which every* ***atom*** *is* ***functional****, and let* $\mathbf{U} = (At\mathbf{A}, ;, I, {}^{\smile})$ *where* $I = {\downarrow}1'$.

Then $\mathbf{U}$ *is a* ***generalized Brandt groupoid*** *(= a category in which each morphism is an isomorphism),* $\mathbf{A}$ *is isomorphic to a* ***subalgebra*** *of the* ***complex algebra*** *of* $\mathbf{U}$ *and* $\mathbf{A}$ *is* ***representable by binary relations****.*

If $\mathbf{A}$ *is* ***simple*** *then* $\mathbf{U}$ *is a* ***Brandt groupoid****.*

If $1'$ *is an* ***atom*** *then* $\mathbf{U}$ *is a* ***group****.*

1952: Time for some lattice theory

$(A, \cdot)$ is a **(meet-)semilattice** if $\cdot$ is **associative**, **commutative** and **idempotent** ($xx = x$)

Define $x \leq y$ if $x \cdot y = x$. It follows that $\leq$ is a partial order and $x \cdot y$ is the **greatest lower bound** (g.l.b. or **inf**) of $\{x, y\}$

Conversely any poset where all pairs have a g.l.b. is a meet-semilattice

$(A, +, \cdot)$ is a **lattice** if $(A, +)$ and $(A, \cdot)$ are semilattices and $x = xy \Leftrightarrow x + y = y$

Equivalently $(A, +, \cdot)$ is a **lattice** if $+, \cdot$ are associative commutative binary operations that satisfy absorption: $x + xy = x = x(x + y)$

$(A, \leq)$ is a **complete lattice** if every $S \subseteq A$ has a g.l.b. $\prod S$ $\quad (\prod \emptyset = 1)$
or equivalently if every $S \subseteq A$ has a **least upper bound** $\sum S$ $\quad (\sum \emptyset = 0)$

Example: $(Eq(U), \subseteq)$, all **equivalence relations** on a set U $\;(\prod S = \bigcap S)$

Bjarni **definitely preferred** $+, 0, \sum, \cdot, 1, \prod$ over $\vee, \perp, \bigvee \wedge, \top, \bigwedge$

On the representation of lattices

B. Jónsson, *Mathematica Scandinavica*. **1**, 193–206, 1953, **5th most cited**

By a **representation of a lattice A** we mean a pair (F, U) such that U is a set and $F : \mathbf{A} \hookrightarrow (Eq(U), +, \cap)$ is a lattice embedding

Here $\theta + \psi = \theta;\psi \cup \theta;\psi;\theta \cup \theta;\psi;\theta;\psi \cup \cdots$ for **equivalence relations** θ, ψ

P. Whitman [1946] showed that every lattice has a representation

(F, U) is **Type 1** if $x, y \in \mathbf{A}$ implies $F(x) + F(y) = F(x); F(y)$

It is **Type 2** if $x, y \in \mathbf{A}$ implies $F(x) + F(y) = F(x); F(y); F(x)$

It is **Type 3** if $x, y \in \mathbf{A}$ implies $F(x) + F(y) = F(x); F(y); F(x); F(y)$

A lattice is **modular** if $x \leq z$ implies $(x + y)z \leq x + yz$ ($\equiv$ to an identity)

Theorem (Jónsson 1953, Thm 1.2, 3.7, 4.2)

A lattice **A** *has a representation of* ***Type 1*** $\iff$ **A** *is isomorphic to a lattice of* ***commuting equivalence relations****.*

A *has a representation of* ***Type 2*** $\iff$ **A** *is* ***modular****.*

Every lattice has a representation of ***Type 3****.*

On the representation of lattices

Bjarni also proved the following results.

Theorem

Any lattice of normal subgroups of a group is Type 1.

There exists a modular lattice without a Type 1 representation.

In fact he exhibits an identity that is satisfied by all modular lattices of Type 1, answering Problem 27 of Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, 1948].

Ends with some **open problems**: Can the class of Type 1 lattices be defined by equations?

How about the class of lattices of normal subgroups of a group,

or the class of lattices of subgroups of an abelian group?

They have been investigated by Herrmann, Haiman, Palfy and many others

Christian Herrmann, A review of some of Bjarni Jónsson's results on representation of arguesian lattices, Algebra Universalis, **70**, 2013, 163–174

Projective spaces and lattices of subspaces

Frink [1946] defined projective spaces without restrictions on the dimension

A **projective space** is a pair $\mathbf{S} = (S, L)$ consisting of a set S of **points** and a collection L of subsets of S, called **lines**, such that

(P_1) for all points $p \neq q \in S$ there is a **unique line** $\ell \in L$ s. t. $p, q \in \ell$,

(P_2) if a line **intersects two sides of a triangle** in distinct points then it **intersects the third side**.

A subset T of S is a **subspace** if $p \neq q \in T$ implies that the line containing p, q is a subset of T.

$(\mathcal{L}(\mathbf{S}), +, \cap)$ is the **complete lattice** of all subspaces of $\mathbf{S}$

a is an **atom** in a lattice if $x < a$ implies $x = 0$ (the bottom element)

A lattice is **atomistic** if every element is a join of atoms

x is **compact** in a complete lattice $\mathbf{A}$ if for all $C \subseteq A$

$$x \leq \sum C \text{ implies } x \leq \sum F \text{ for some finite } F \subseteq C$$

Modular lattices and Desargues' theorem

B. Jónsson, *Mathematica Scandinavica*. 2, 295–314, 1954

A (semi-)modular lattice is **geometric** if it is atomistic, complete and all atoms are compact

Birkhoff [1935] proved that geometric lattices are **complemented**

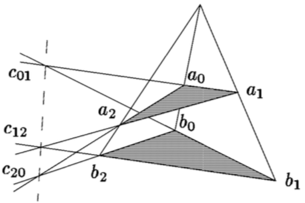
Frink [1946] proved that every **complemented modular lattice M** is embedded in a lattice $\mathbf{M}'$ of **subspaces of some projective space**

Bjarni **generalized** perfect extensions to **complemented modular lattices** and showed that $\mathbf{M}'$ is a perfect extension of **M**, so satisfies the **same lattice identities** as **M**

Theorem (Jónsson 1954, Thm 2.14)

*For a **complemented modular lattice** M the following are equivalent:*
(i) M *has a representation of **Type 1***
(ii) M *is embedded in the lattice of **normal subgroups of a group***
(iii) M *is embedded in the lattice of **subgroups of an abelian group***
(iv) M *is embedded in the lattice of **subspaces of a projective space***
(v) M *is **Arguesian***

Desargues' law and Arguesian lattices



Desargues' law holds in a projective space if for all points $a_0 a_1 a_2 b_0 b_1 b_2$

$$(a_0 + b_0)(a_1 + b_1) \leq a_2 + b_2 \implies c_{01} \leq c_{12} + c_{20}$$

where $c_{ij} = (a_i + a_j)(b_i + b_j)$ for $i \neq j \in \{0, 1, 2\}$

A lattice is **Arguesian** if for $d = c_{01}(c_{12} + c_{20})$ it satisfies the identity

$$(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0(a_1 + d) + b_0(b_1 + d)$$

Desargues' law and Arguesian lattices

Theorem (Jónsson 1953)

*A projective space satisfies **Desargues' law** if and only if its lattice of subspaces is **Arguesian**.*

Theorem (Jónsson 1954)

*Every **Arguesian** lattice is **modular**.*

Theorem (Jónsson, G.S. Monk 1969)

*Every **Arguesian** lattice satisfies **Desargues' law**.*

Lemma (Grätzer, Lakser, Jónsson 1973)

*For any modular lattice, **Desargues' law** implies the **Arguesian** identity.*

Theorem (Jónsson 1972)

*The class of **Arguesian** lattices is **self-dual**.*

The class of **Arguesian** lattices is **self-dual**

Proof.

A modular lattice is **arguesian** iff $(a_0 + b_0)(a_1 + b_1) \leq a_2 + b_2$ (1)

$$\implies (a_0 + a_1)(b_0 + b_1) \leq (a_0 + a_2)(b_0 + b_2) + (a_1 + a_2)(b_1 + b_2) \quad (2)$$

The **dual** implication is $x_0 y_0 + x_1 y_1 \geq x_2 y_2$ (3)

$$\implies x_0 x_1 + y_0 y_1 \geq (x_0 x_2 + y_0 y_2)(x_1 x_2 + y_1 y_2) \quad (4)$$

Take $a_0 = x_0 x_2, a_1 = y_0 y_2, a_2 = x_0 y_0, b_0 = x_1 x_2, b_1 = y_1 y_2, b_2 = x_1 y_1$.

Easy calculations show that (3)⇒(1) and (2)⇒(4).

Consequently if (1)⇒(2) then (3)⇒(4). □

Moving from Rhode Island to Minnesota

At **Brown University** (1946-1956), Bjarni also published results about

Distributive sublattices of modular lattices, *Proc. Amer. Math. Soc.* **6** (5), 682–688, 1955

and about model theory: An appendix to Tarski's book **Ordinal Algebras** on **A unique decomposition theorem for relational addition**

and on **Universal relational systems**, *Math. Scand.* **4**, 193–208, 1957

where he shows that lattices have the **strong amalgamation property**.

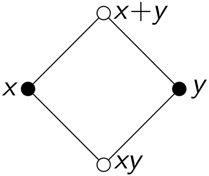
At the **University of Minnesota** he is working on **isomorphism types of algebraic systems**, **direct decompositions of torsionfree abelian groups**, **homogeneous universal relational systems**, and **the representation of modular lattices and relation algebras**, ...

Sublattices of a free lattice

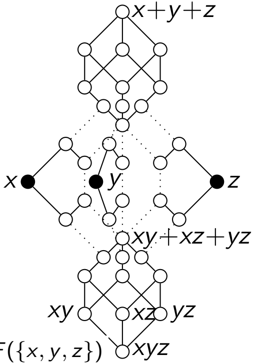
20. B. Jónsson, *Canad. J. Math.* **13**, 256–264, 1961

The **free lattice** $F(X)$ is the "**most general lattice**" that can be generated from an unordered set X of generators.

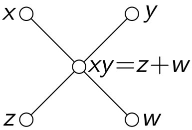
$F(\{x\})$



$F(\{x, y\})$



$F(\{x, y, z\})$



does not occur

Whitman [1941] showed that free lattices satisfy

(W) $xy \le z + w \implies xy \le z$ or $xy \le w$ or $x \le z + w$ or $y \le z + w$

Sublattices of a free lattice

Bjarni proved that free lattices are **semidistributive**, i.e. they satisfy

(SDm) $xy = xz \implies x(y+z) = xy$ and

(SDj) $x + y = x + z \implies x + yz = x + y$

These 3 **universal** properties hold in $\mathcal{F} = \{$**sublattices of free lattices**$\}$

Is every **finite lattice** that satisfies **(W), (SDm), (SDj)** in $\mathcal{F}$?

In the current paper he proves that $\mathcal{F}$ is closed under coproducts.

More generally, let $\mathcal{V}$ be a variety that has the **joint embedding property** ($A, B \in \mathcal{V}$ implies $A, B \hookrightarrow C \in \mathcal{V}$) and the **amalgamation property** ($f : A \hookrightarrow B \in \mathcal{V}, g : A \hookrightarrow C \in \mathcal{V}$ implies $\exists f' : B \hookrightarrow D, g' : C \hookrightarrow D \in \mathcal{V}$ s.t. $f' \circ f = g' \circ g$).

Theorem (1.3, 1.4, Jónsson 1961)

The class of ***subalgebras*** *of* $\mathcal{V}$*-****free algebras*** *is closed under* ***coproducts***

Sublattices of a free lattice

The **height** of a lattice L is $\sup\{|C|-1 : C$ is a subchain of $L\}$

Theorem (2.7, Jónsson 1961)

*A **finite-height** sublattice of a **meet-semidistributive** lattice is **finite**.*

Proof.

Assume $L \models$ (SDm), has height n and the result holds for all lattices with height $< n$.
Let M be the set of atoms of L, choose $a \in M$ and let $N = M - \{a\}$.
Then $ab = 0$ for all $b \in N$.
Defining $c = \sum N$, it follows from (SDm) that $ac = 0$.
Hence $c \neq 1$, so $N \subseteq {\downarrow}c$ is finite and therefore M is finite.
By hypothesis ${\uparrow}d$ is finite for all $d \in M$, hence L is finite. □

Corollary

*Any **finite-height sublattice** of a **free lattice** is **finite**.*

Finite sublattices of a free lattice

25. B. Jónsson, J. E. Kiefer, *Canad. J. Math.* **14**, 487–497, 1962

If $a = \sum U = \sum V$ then U **refines** V if $\forall u \in U\ \exists v \in V \quad u \leq v$

$a = \sum U$ is a **canonical join-representations** if it is **irredundant** (no proper subset joins to a) and U refines **every other join-representation**.

This **generalizes** Whitman's notion of canonical representation in a free lattice

Theorem (Jónsson and Keifer, 1962)

*(SDj) implies any two join-representations have a **common refinement**, so in a **finite join-semidistributive lattice** every element has a **canonical join-representation**.*
*Conversely if **every element** in a lattice has a **canonical join-representation** then (SDj) holds.*

Hence in a **finite semidistributive lattices** every element has a **canonincal join- and meet-representation**

Subdirectly irreducible algebras and ultraproducts

An algebra $\mathbf{A}$ is a **subdirect product** of $\{\mathbf{A}_i : i \in I\}$ if $h : \mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}_i$ is an embedding and $\pi_i \circ h$ is surjective for all $i \in I$

$\mathbf{A}$ is **subdirectly irreducible** if $\mathrm{Con}(\mathbf{A}) \setminus \{id_A\}$ has a **smallest** element

$\mathcal{V}_{SI}$ is the class of **subdirectly irreducible members** of a variety $\mathcal{V}$

Theorem (Birkhoff 1944, using the Axiom of Choice)

Every algebra $\mathbf{A}$ *is a* ***subdirect product*** *of* ***subdirectly irreducible homomorphic images*** *of* $\mathbf{A}$, *hence* $\mathcal{V} = SP(\mathcal{V}_{SI})$

So the subdirectly irreducible algebras are **building blocks** of varieties

An **ultrafilter** of a lattice is a maximal up-closed proper sublattice

For a set I, let $\mathcal{U}$ be an ultrafilter of $\mathcal{P}(I)$

The **ultraproduct** $\prod_{\mathcal{U}} \mathbf{A}_i$ is $(\prod_{i\in I} \mathbf{A}_i)/\theta_{\mathcal{U}}$ where $\theta_{\mathcal{U}}$ is the **congruence** $\{(x,y) : \{i \in I : x_i = y_i\} \in \mathcal{U}\}$

$P_u\mathcal{K}$ is the **class of all ultraproducts** of members of $\mathcal{K}$

Algebras whose congruence lattices are distributive

33. B. Jónsson, Mathematica Scandinavica. 21, 110–121, 1967

7. Added April 1967.

A preliminary account of the work presented here was prepared and privately circulated during the winter 1963–64, and the results were reported at the 1964 Congress of Scandinavian Mathematicians. This manuscript was submitted in September 1964, and was accepted for publication in the Proceedings of the Congress. Because of a delay in the publication of these Proceedings, the manuscript was transferred to Mathematica Scandinavica in February 1967, and accepted for publication in April 1967.

MR Review: The importance of the two main results was immediately recognized by many mathematicians. As a result of this a large number of papers have since been written and published influenced by these results.

The paper appeared exactly 50 years ago

It has been cited 604 times according to Google Scholar