On the enduring impact of Bjarni Jónsson's research in Lattice Theory, Relation Algebras and Boolean Algebras with Operators, Part 2

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BLAST: A conference in memory of Bjarni Jónsson

Outline

- Congruence distributivity 1963-1967
- Varieties of lattices 1968-1979
- Arithmetic of ordered sets 1980-1982
- The theory of binary relations 1982-1994
- BAOs and distributive lattices expansions 1993-2004

Algebras whose congruence lattices are distributive

33. B. Jónsson, Mathematica Scandinavica. 21, 110–121, 1967

7. Added April 1967.

A preliminary account of the work presented here was prepared and privately circulated during the winter 1963–64, and the results were reported at the 1964 Congress of Scandinavian Mathematicians. This manuscript was submitted in September 1964, and was accepted for publication in the Proceedings of the Congress. Because of a delay in the publication of these Proceedings, the manuscript was transferred to Mathematica Scandinavica in February 1967, and accepted for publication in April 1967.

MR Review: The importance of the two main results was immediately recognized by many mathematicians. As a result of this a large number of papers have since been written and published influenced by these results.

The paper appeared exactly 50 years ago

Algebras whose congruence lattices are distributive

A is **congruence distributive** (CD) if Con(**A**) is a distributive lattice

A class $\mathcal{K}$ of algebras is **CD** if every algebra in $\mathcal{K}$ is **CD**

Theorem (Jónsson 1967)

*A variety $\mathcal{V}$ is **CD** if and only if there exist terms $t_0, \ldots, t_n$ $(n \geq 2)$ s. t.*

$$t_0(x,y,z) = x \qquad t_i(x,y,x) = x \qquad t_n(x,y,z) = z$$

$$t_i(x,x,z) = t_{i+1}(x,x,z) \text{ for even } i, \;\; t_i(x,z,z) = t_{i+1}(x,z,z) \text{ for odd } i$$

This Mal'cev condition is referred to as **Jónsson terms**

Proof outline. (⇒).

For $\mathbf{A} = \mathbf{F}_{\mathcal{V}}\{a,b,c\}$, $(a,c) \in \theta_{a,c} \cap (\theta_{a,b} + \theta_{b,c}) = (\theta_{a,c} \cap \theta_{a,b}) + (\theta_{a,c} \cap \theta_{b,c})$
Hence for some $n \geq 2$ there exist $d_0, \ldots, d_n \in A$ such that
$a = d_0 \, (\theta_{a,c} \cap \theta_{a,b}) \, d_1 \, (\theta_{a,c} \cap \theta_{b,c}) \, d_2 \, (\theta_{a,c} \cap \theta_{a,b}) \, \ldots \, d_n = c$.
Since **A** is generated by a, b, c there are terms $t_i(a,b,c) = d_i$.
$a \theta_{a,c} \, t_i(a,b,c) \, \theta_{a,c} \, t_i(a,b,a)$ and **A** free implies $x = t_i(x,y,x)$.
For i even, $t_i(a,a,c) \, \theta_{a,b} \, t_i(a,b,c) \, \theta_{a,b} \, t_{i+1}(a,b,c) \, \theta_{a,b} \, t_{i+1}(a,a,c)$
A free implies $t_i(x,x,y) = t_{i+1}(x,x,y)$, and similarly for i odd. □

Algebras whose congruence lattices are distributive

Proof outline. ($\Leftarrow$).

Let $\mathbf{A} \in \mathcal{V}$, $\quad \theta, \varphi, \psi \in \mathrm{Con}(\mathbf{A})$ and $(a, c) \in \theta \cap (\varphi + \psi)$.
Then $(a, c) \in \theta$ and $a = b_0 \,\varphi\, b_1 \,\psi\, b_2 \,\varphi\, b_3 \,\psi \cdots b_m = c$ for some b_i.
$\Rightarrow t_i(a, b_0, c) \,\varphi\, t_i(a, b_1, c) \,\psi\, t_i(a, b_2, c) \,\varphi \cdots t_i(a, b_m, c)$ for all i.
From $t_i(x, y, x) = x$ and $(a, c) \in \theta$ we deduce
$t_i(a, b_0, c) \,\theta{\cap}\varphi\, t_i(a, b_1, c) \,\theta{\cap}\psi\, t_i(a, b_2, c) \,\theta{\cap}\varphi \cdots t_i(a, b_m, c)$
hence $t_i(a, a, c) \,(\theta \cap \varphi) + (\theta \cap \psi)\, t_i(a, c, c)$ for all i.
The remaining identities now imply $(a, c) \in (\theta \cap \varphi) + (\theta \cap \psi)$. □

Note: $t_0 = x, t_1 = xy + xz + yz, t_2 = z$ are **Jónsson terms** for **lattices**.

Hence lattices are a **CD** variety; same for any variety with lattice reducts.

This had been proved for lattices by Funayama and Nakayama [1942]

Now the **MAIN Lemma**.

Quote: "The proof is quite simple, but some of the consequences are rather unexpected." Recall $\theta_{\mathcal{U}} = \{(x, y) : \{i \in I : x_i = y_i\} \in \mathcal{U}\}$.

Jónsson's Lemma 1967 (but really 1963)

Lemma (3.1.)

If $\mathbf{A}$ *is a* ***CD*** *subalgebra of* $\prod_{i\in I}\mathbf{A}_i$, $\varphi\in Con(\mathbf{A})$ *and* $\mathbf{A}/\varphi$ *is* ***subdirectly irreducible*** *then there exists an ultrafilter* $\mathcal{U}$ *such that* $\theta_{\mathcal{U}}\cap A^2\subseteq\varphi$

Proof.

For $J\subseteq I$ define $\hat{J}=\theta_{[J)}\cap A^2$ where $[J)=\{K\subseteq I : J\subseteq K\}$.
Let $\mathcal{D}=\{J : \hat{J}\subseteq\varphi\}$. Then $\emptyset\notin\mathcal{D}$, $I\in\mathcal{D}$, and $K\supseteq J\in\mathcal{D}\implies K\in\mathcal{D}$.
(*) If $J\cup K\in\mathcal{D}$ then $\varphi=\varphi+\widehat{J\cup K}=\varphi+(\hat{J}\cap\hat{K})=(\varphi+\hat{J})\cap(\varphi+\hat{K})$.
$\mathbf{A}/\varphi$ subdirectly irred. implies φ **meet irred.**, hence $J\in\mathcal{D}$ or $K\in\mathcal{D}$.
By Zorn's Lemma, let $\mathcal{U}$ be a maximal subset of $\mathcal{D}$ that is a filter of $\mathcal{P}(I)$.
Then $\theta_{\mathcal{U}}\cap A^2=\bigcup_{J\in\mathcal{U}}\hat{J}\subseteq\psi$, as required.
Suppose $\mathcal{U}$ is not an ultrafilter. Then there exists $H\subset I$ s. t. $H, I\setminus H\notin\mathcal{U}$.
$H\cap J\notin\mathcal{D}$ for some $J\in\mathcal{U}$, else $\mathcal{U}\cup\{H\}$ generates a filter in $\mathcal{D}$.
Similarly $(I\setminus H)\cap K\notin\mathcal{D}$ for some $K\in\mathcal{U}$. But then $J\cap K\in\mathcal{U}$ so
$J\cap K\subseteq(H\cap J)\cup((I\setminus H)\cap K)\in\mathcal{D}$, contradicting (*). □

Algebras whose congruence lattices are distributive

Łos 1955: If $\mathcal{K} \models \sigma$ then $P_u\mathcal{K} \models \sigma$ for any first order formula σ

Frayne, Morel, Scott 1962: For a finite class $\mathcal{K}$ of finite algebras $P_u\mathcal{K} = \mathcal{K}$

Corollary (Jónsson 1967)

*If $\mathcal{V} = V(\mathcal{K})$ is **congruence distributive** then $\mathcal{V}_{SI} \subseteq HSP_u\mathcal{K}$*

*If $\mathcal{K}$ is a **finite** class of **finite** algebras and $V(\mathcal{K})$ is **CD** then $\mathcal{V}_{SI} \subseteq HS\mathcal{K}$*

*If $\mathbf{A}, \mathbf{B} \in \mathcal{V}_{SI}$ are **finite nonisomorphic** and $\mathcal{V}$ is **CD** then $V(\mathbf{A}) \neq V(\mathbf{B})$*

Proof.

Since $\mathcal{V} = V(\mathcal{K}) = HSP\mathcal{K}$, for every $\mathbf{B} \in \mathcal{V}_{SI}$ there exist a subalgebra $\mathbf{A}$ of a product $\prod_{i \in I} \mathbf{A}_i$ and $\varphi \in \mathrm{Con}(\mathbf{A})$ such that $\mathbf{B} \cong \mathbf{A}/\varphi$.
By Jónsson's Lemma there exists an ultrafilter $\mathcal{U}$ such that $\theta_{\mathcal{U}} \cap A^2 \subseteq \varphi$.
Then $\mathbf{A}/\varphi$ is a homomorphic image of $\mathbf{A}/(\theta_{\mathcal{U}} \cap A^2)$ which is a subalgebra of the ultraproduct $\prod_{\mathcal{U}} \mathbf{A}_i$, hence $\mathbf{B} \in HSP_u\mathcal{K}$.
If $\mathbf{A} \ncong \mathbf{B}$ are finite s.i. and $|\mathbf{A}| \leq |\mathbf{B}|$ then $\mathbf{B} \notin HS\{\mathbf{A}\}$. □

Algebras whose congruence lattices are NOT distributive

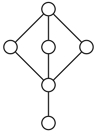
If an algebra $\mathbf{A}$ is CD, it does NOT follow that $V(\mathbf{A})$ is CD.

E.g. in groups, $\mathrm{Con}(\mathbb{Z}_2) \cong \mathbf{2}$ but $\mathrm{Con}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbf{M}_3$.

Groups show that Jónsson's Lemma does **not** apply to congruence **modular** or congruence **permutable** or congruence **Arguesian** varieties.

Let $\mathbf{D}_4 = \langle a, b : a^4 = 1 = b^2, ba = a^3 b\rangle$ be the **dihedral group** of size 8

Let $\mathbf{Q}_8 = \langle i, j : i^4 = 1, i^2 = j^2, ji = i^3 j\rangle$ be the **quaternion group**.

They are nonisomorphic and subdir. irred. (Cong. lattice = )

Exercise: show $\mathbf{D}_4 \in HS(\mathbf{Q}_8 \times \mathbf{Q}_8)$ and $\mathbf{Q}_8 \in HS(\mathbf{D}_4 \times \mathbf{D}_4)$

so $V(\mathbf{D}_4) = V(\mathbf{Q}_8)$, i.e. they generate the **same variety of groups**.

Named lemmas (from Wikipedia)

Abel, Abhyankar, Archimedes, Barbalat, Berge, Bézout, Bhaskara, Blichfeldt, Borel, Burnside, Céa, Couchman, Cousin, Craig, Dehn, Dickson, Dobrushin, Dwork, Dynkin, Ehrling, Euclid, Farkas, Fatou, Feinstein, Fekete, Finsler, Fitting, Fodor, Frattini, Friedrichs, Frostman, Gauss, Gödel, Goursat, Grönwall, Gromov, Gross, Grothendieck, Haruki, Hartogs, Hayashi, Hensel, Higman, Hopf, Hotelling, Hua, Huet, Itō, **Jónsson**, Jordan, Kelly, Klop, Knuth, König, Kronecker, Krull, Lebesgue, Lindelöf, Lindenbaum, Lions, Little, Łojasiewicz, Lovász, Margulis, Mautner, Morse, Moschovakis, Mostowski, Nakayama, Newman, Noether, Ogden, Poincaré, Pugh, Racah, Ricci, Riesz, Robbins, Sard, Schanuel, Schreier, Schur, Schwarz, Shephard, Siegel, Sperner, Stechkin, Stein, Szemerédi, Urysohn, Varadhan, Vaughan, Vitali, Vizing, Wald, Watson, Weyl, Whitehead, Yao, Yoneda, Zassenhaus, Zolotarev, Zorn **(103 lemmas with one name)**

Artin–Rees, Aubin–Lions, Borel–Cantelli, Bramble–Hilbert, Brezis–Lions, Cotlar–Stein, Danielson–Lanczos, Davis–Figiel–Johnson–Pelczynski, Deny–Lions, Doob–Dynkin, Ellis–Numakura, Feld–Tai, Glivenko–Cantelli, Hardy–Littlewood, Hindley–Rosen, Johnson–Lindenstrauss, Kalman–Yakubovich–Popov, Knaster–Kuratowski–Mazurkiewicz, Kuratowski–Zorn, Littlewood–Offord, Neyman–Pearson, Pólya–Burnside, Rasiowa–Sikorski, Riemann–Lebesgue, Rouche–Kronecker–Campelli, Lax–Milgram, Schwartz–Zippel, Stewart–Walker, Teichmüller–Tukey, Verdu–Han **(30 lemmas with 2 or more names)**

There are **920 named theorem** listed in Wikipedia

Conclusion: $\ddot\smile$

A **named lemma** is $\frac{920}{133} \approx 7$ times **more special** than a **named theorem**

Lattices of subvarieties

$\mathcal{V}$ is **finitely generated** if $\mathcal{V} = V(\mathcal{K})$ for a finite class $\mathcal{K}$ of finite algebras

Corollary (Jónsson 1967)

*A finitely generated **CD** variety has only **finitely** many subvarieties*

For a variety $\mathcal{V}$ the **lattice of subvarieties** is denoted by $\Lambda_{\mathcal{V}}$

The meet is $\bigcap$ and the join is $\sum_{i \in I} \mathcal{V}_i = V(\bigcup_{i \in I} \mathcal{V}_i)$

Theorem (Jónsson 1967)

$HSP_u(\mathcal{K} \cup \mathcal{L}) = HSP_u\mathcal{K} \cup HSP_u\mathcal{L}$ *for any classes* $\mathcal{K}, \mathcal{L}$.

If $\mathcal{V}$ *is **CD** then* $\Lambda_{\mathcal{V}}$ *is distributive.*

As mentioned before, the variety of lattices is CD, so Bjarni now used his fundamental results to analyze the lattice of equational classes of lattices.

Varieties of Lattices

The variety $\mathcal{L}$ of **all lattices** is the **largest lattice variety**.

The **smallest lattice variety** is $\mathcal{O} = \{$one-element lattices$\}$.

The variety $\mathcal{D} = V(\mathbf{2})$ of **distributive** lattices is the **only atom** of $\Lambda_{\mathcal{L}}$.

Birkhoff [1935]: $\mathbf{M}_3$ is a sublattice of every **nondistributive modular lattice**.

Dedekind [1900]: every **non-modular lattice** contains $\mathbf{N}_5$ as a sublattice.

$\implies$ $\mathcal{M}_3 = V(\mathbf{M}_3)$ and $\mathcal{N}_5 = V(\mathbf{N}_5)$ are the only covers of $\mathcal{D}$

Moving to Vanderbilt University, Nashville, Tennessee

In 1966 Bjarni is hired as the **first distinguished professor** at Vanderbilt.

He is working on **model theory**, **almost direct products and saturation**, **relatively free lattices**, **varieties of lattices**, **finitely based varieties**,...

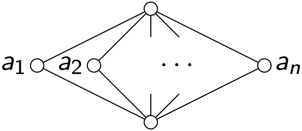
and is writing a book **Topics in Universal Algebra** (Lecture Notes in Math. 250, Springer Verlag, vi+220 pp, 1972)

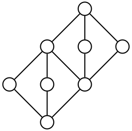
The first issue of **Algebra Universalis** appears, including a paper by Bjarni on **Relatively free products of lattices**, Algebra Universalis **1** (1), 362–373, 1971

In 1974 Bjarni is one of the invited speakers at the **International Congress of Mathematicians** in Vancouver together with other notable mathematicians, e.g., V. I. Arnold, Hyman Bass, Alain Connes, Pierre Delinge, E. B. Dynkin, Charles Fefferman, Harvey Friedman, András Hajnal, Horst Herrlich, Victor Klee, Daniel Kleitman, Barry Mazur, Yiannis Moschovakis, Louis Nirenberg, Daniel Quillen, Richard Rado, H. L. Royden, Mary Ellen Rudin, Saharon Shelah, Barry Simon, William Thurston, John Wilder Tukey, ...

Equational classes of lattices

34. B. Jónsson, *Math. Scand.* **22** (1), 187–196, 1968

Let $\mathbf{M}_n =$ 

and $\mathbf{M}_{3,3} =$ 

Theorem

For a variety $\mathcal{V}$ of modular lattices, $\mathbf{M}_{3,3} \notin \mathcal{V}$ if and only if $\mathcal{V}_{SI}$ only contains lattices of height 2 or less

Note that a lattice of height 2 is isomorphic to $\mathbf{M}_\kappa$ for some cardinal κ.

It follows that $V(\mathbf{M}_4)$ and $V(\mathbf{M}_{3,3})$ are the only covers of $V(\mathbf{M}_3)$.

Bjarni also showed that the varieties $V(\mathbf{M}_n)$ form a **covering chain**,

$V(\mathbf{M}_\omega) = V(\mathbf{M}_\kappa)$ for **any infinite** cardinal κ, and

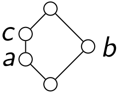
he found **equational bases** for these varieties.

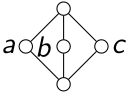
Critical edges in subdirectly irreducible lattices

55. B. Jónsson, I. Rival, *Proc. Amer. Math. Soc.* **66** (2), 194–196, 1977

This paper expands on the classical results of **Dedekind** and **Birkhoff**.

Define $N(a,b,c) \iff bc < a < c < a+b$





$M(a,b,c) \iff a+b=b+c=c+a$ and $ab=bc=ca$.

(u,v) is **critical** in a s.i. lattice $\mathbf{L}$ if $u<v$ and $u\theta v$ for all $id_L \neq \theta \in \mathrm{Con}(\mathbf{L})$

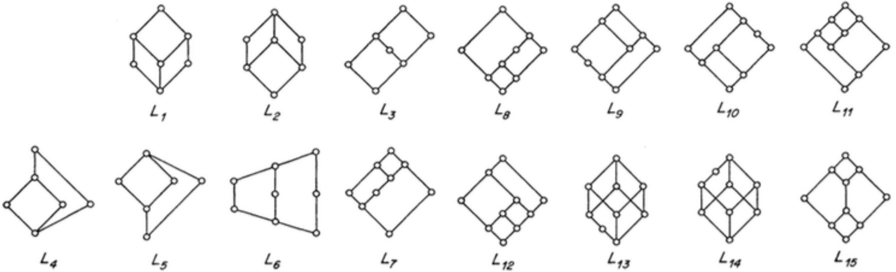
Theorem

*(i) Every **nonmodular subdirectly irreducible** lattice contains a **critical** pair (a,c) for some $b \in \mathbf{L}$ with $N(a,b,c)$.*
*(ii) Every **nondistributive modular subdirectly irreducible** lattice contains a **critical** pair $(a,a+b)$ or (ab,a) such that $M(a,b,c)$.*

Lattice varieties covering the smallest nonmodular variety

56. B. Jónsson, I. Rival, *Pacific J. Math.* **82** (2), 463–478, 1979

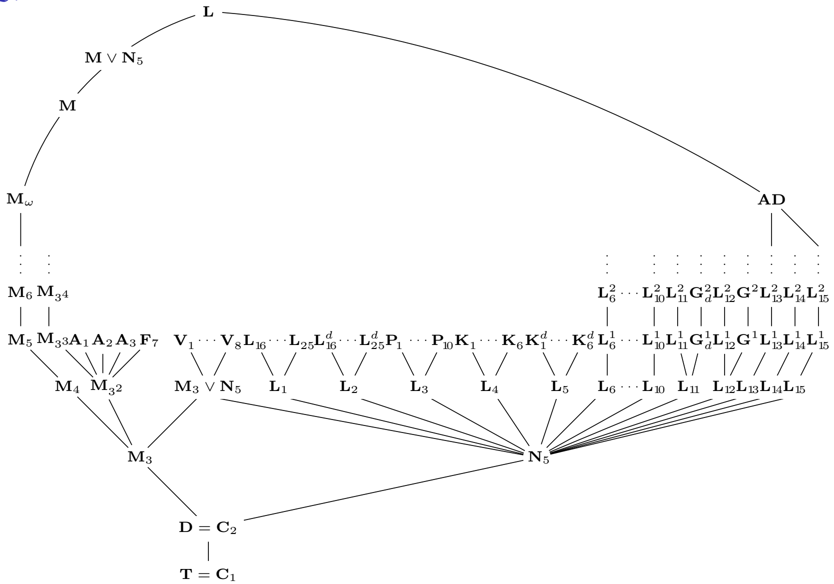
McKenzie [1970] found 15 s. i. lattices that generate varieties covering $\mathcal{N}_5$



Theorem (Jónsson and Rival, 1979)

Every variety of lattices that properly contains $\mathcal{N}_5$ includes one of the lattices $M_3, L_1, L_2, \ldots, L_{15}$.

$\Lambda_{\mathcal{L}}$, the Lattice of Varieties of Lattices



Based on results by B. Jónsson '68, R. McKenzie '70, D. X. Hong '72, W. Ruckelshausen '78, H. Rose '84, J.G. Lee '85, J.B. Nation '86, C.Y. Wong '89

Varieties of lattices: Some open problems

61. B. Jónsson, *Colloq. Math. Soc. János Bolyai* **29**, 421-436, 1982

"Good problems can be **extremely important** in stimulating research,..."

The paper provides background and discussion of **54 problems** in 8 sections: the variety $\mathcal{D}$ (4), the variety $\mathcal{L}$ (7), the variety $\mathcal{M}$ (12), the Arguesian identity (2), other varieties of lattices (5), the lattice Λ (8), equational bases (4), congruence varieties (12).

Many (but not all) of these problems have now been solved. E.g.,

Problem 3.1. Does there exist a finite set of first order formulas whose finite models are precisely the **finite sublattices of free lattices**? In particular, do (W), (SDm), (SDj) form such a set? **YES**. J.B. Nation, Finite sublattices of a free lattice. *Trans. AMS.* **269** (1982), no. 1, 311–337

Problem 4.3. Find an identity that holds in **every finite modular** lattice but **not in all modular** lattices. (open)

Powers of partially ordered sets: Cancellation and refinement

62. **B. Jónsson, R. McKenzie,** *Math. Scand.* **51** (1), 87–120, 1982

Given posets A, B let $A^B = \{f : B \to A \mid x \leq y \Rightarrow f(x) \leq f(y)\}$

A^B is again a poset ordered **pointwise**: $f \leq g \iff \forall x \in B\ f(x) \leq g(x)$

This exponentiation operation was studied by Birkhoff [1937, 1942], Novotny [1960] and Duffus and Rival [1978]

Let $A + B =$ **disjoint union** and $A \cdot B =$ **direct product** (also written AB)

The following **hold**: $A^{B+C} \cong A^B \cdot A^C$, $(A^B)^C \cong A^{B \cdot C}$, $(AB)^C \cong A^C \cdot B^C$

For **unordered** sets, $+, \cdot, A^B$ are the **cardinal arithmetic** operations.

For **finite unordered** sets they are the **standard artihmetic** operations.

Interestingly, if C is **connected** then $(A + B)^C \cong A^C + B^C$

Powers of partially ordered sets: Cancellation and refinement

What about: I. Cancellation for **bases**. $A^B \cong A^C \Rightarrow B \cong C$

II. Cancellation for **exponents**. $A^C \cong B^C \Rightarrow A \cong B$

III. Refinement for **powers**. $A^C \cong B^D \Rightarrow A \cong E^X, B \cong E^Y, C \cong YZ$, $D \cong XZ$ for some E, X, Y, Z, hence both sides are isomorphic to E^{XYZ}

IV. **Mixed** refinement. $A^B \cong \prod_{i\in I} C_i \Rightarrow A \cong \prod_{i\in I} A_i$ and $C_i \cong A_i^B$ for all i

The paper contains examples showing each of I-IV. **can fail**.

A poset is **finitely factorable** if it has a unique factorization into finitely many indecomposables. The paper contains many results, e. g.

Theorem (Jónsson and McKenzie, 1982)

Refinement of powers *(III.) holds if*

*(i) A^C is **atomic** and C, D are **finitely factorable** and upper bounded **or***

*(ii) A, B are **meet-semilattices** and C, D are **finite** and upper bounded.*

*In either case, if $C \cong D$ then **cancellation of exponents** (II.) holds.*

Powers of partially ordered sets: the automorphism group

63. B. Jónsson, *Math. Scand.* **51** (1), 121-141, 1982

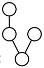
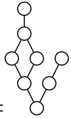
For a poset P the **automorphism group** is denoted $\mathrm{Aut}(P)$.

P is **exponentially indecomposable** if $P \ncong X^Y$ for all posets X, Y with $|Y| > 1$

Theorem (Jónsson, 1982)

Suppose P is a bounded directly indecomposable poset that satisfies the descending chain condition, and suppose $P \cong A^C$ where A is exponentially indecomposable. Then $Aut(P) \cong Aut(A) \times Aut(C)$.

There is a natural embedding $\pi : \mathrm{Aut}(A) \times \mathrm{Aut}(C) \hookrightarrow \mathrm{Aut}(A^C)$ defined by $\pi(\alpha, \gamma)(f) = \alpha \circ f \circ \gamma^{-1}$.
The proof shows that π is an isomorphism under the stated assumptions.

$A =$  and $C =$  have trivial aut. group, but $A^C =$  so $|\mathrm{Aut}(A^C)| = 2$

Arithmetic of ordered sets

64. B. Jónsson, *Banff 1981, NATO Adv. Sci. Inst. Ser. C* **83**, 3-41, 1982

This is a **survey paper** about operations on partially ordered sets.

Many of the results also apply to more general relational structures.

Bjarni considers the operations $+, \cdot$ and exponentiation of posets, but now also adds **ordinal sum** $\oplus$, **lexicographic product** $\circ$ and **converse** $^{\smile}$.

The **algebra** of these 6 operations, as well as **cancellation** and **refinement properties** are discussed in detail.

Varieties of relation algebras

65. B. Jónsson, *Algebra Universalis* **15** (1), 273–298, 1982

This is another example of a delightful **survey paper**.

More than 40 years of in-depth studies on **abstract relation algebras** are summarized in an accessible way in a mere **26 pages**.

All the **basic results**, **relativized relation algebras**, **atom structures**,

examples of small **non-representable relation algebras**,

complex algebras of group(oid)s, connections to **modular lattices**,

discriminator varieties, **representable relation algebras are a variety**,

the **lattice of subvarieties**, its **atoms**, **coatoms** and **center**,

equational bases, **splitting algebras** and **conjugate varieties**, number of **subvarieties**, . . .

Maximal algebras of binary relations

66. B. Jónsson, *Contemp. Math.* **33**, 299–307, 1984

Let $\mathcal{R}(X) = (\mathcal{P}(X^2), \cup, \cap, ^{-}, \circ, ^{\smile}, id_X)$ be the full representable RA and

$\Pi(X) = (S_X, \circ, ^{-1}, id_X) \cong \mathrm{Aut}(R(X)) : \phi \mapsto \phi^*$ where $\phi^*(R) = \phi^{-1} \circ R \circ \phi$

Then there is a **Galois connection**

$A^\sigma = \{R \in \mathcal{R}(X) : \phi^*(R) = R \text{ for all } \phi \in A\}$ for $A \subseteq \Pi(X)$

$\mathcal{S}^\rho = \{\phi \in \Pi(X) : \phi^*(R) = R \text{ for all } R \in \mathcal{S}\}$ for $\mathcal{S} \subseteq \mathcal{R}(X)$

Theorem (Jónsson 1984)

If G is a subgroup of $\Pi(X)$ of prime order, then G^σ is a maximal proper subalgebra of $\mathcal{R}(X)$.

The Galois connection is used to calculate all subalgebras of $\mathcal{R}(4)$ (29) and $\mathcal{R}(5)$ (124)

Relation algebras and Schröder categories

74. B. Jónsson, *Discrete Math.* **70** (1), 27–45, 1988

A **Schröder category** is an **enriched** category C in which the set of **morphisms** $C(i,j)$ between any two objects i,j is a **Boolean algebra**, there is a contravariant endofunctor $^{\smile}$ on C taking $C(i,j)$ to $C(j,i)$, and for all $x \in C(i,j)$, $y \in C(j,k)$ and $z \in C(i,k)$

$$(x;y)z = 0 \iff (x^{\smile};z)y = 0 \iff (z;y^{\smile})x = 0.$$

These categories are used to construct a **semi-product** which combines a family of **simple relation algebras** into a **big simple relation algebra** having the components as **relative subalgebras**.

An element x is an equivalence element if $x;x = x = x^{\smile}$.

Theorem

*Every relation algebra that is **generated** by an **equivalence element** is **finite and representable**.*

Relation algebras as residuated Boolean algebras

82. B. Jónsson, C Tsinakis, *Algebra Universalis* **30** (4), 469–478, 1993

$\mathbf{A} = (\mathbf{A}_0, \circ, \triangleright, \triangleleft)$ is a **residuated Boolean algebra** (r-algebra for short) if $\mathbf{A}_0 = (A, +, \cdot, ^-, 0, 1)$ is a **Boolean algebra** and
$(x \circ y)z = 0 \iff (x \triangleright z)y = 0 \iff (x \triangleleft z)y = 0$

If $\circ$ has a **unit** e (i.e. $x \circ e = x = e \circ x$) then $(\mathbf{A}_0, \circ, e, \triangleright, \triangleleft)$ is called a *ur*-algebra.

Among other results, the paper shows that relation algebras are (term-equivalent to) a finitely based subvariety of *ur*-algebras:

Theorem (Jónsson and Tsinakis, 1993)

A ur-algebra $\mathbf{A}$ *is a relation algebra (with* $x^{\smile} = x \triangleright e$*) if and only if* $x \triangleright (y \circ z) = (x \triangleright y) \circ z$.

These investigations ultimately lead to extensive research on residuated lattices.

Adjoining units to residuated Boolean algebras

86. P. Jipsen, B. Jónsson, J. Rafter, *Algebra Univ.* **34** (1), 118–127, 1995

A **subreduct** $\mathbf{A} = (\mathbf{A}_0, \circ, ^\smile)$ of a relation algebra $(\mathbf{A}_0, \circ, e, ^\smile)$ is a subalgebra of the e-free reduct.

Tarski had conjectured that the set of **identities holding in all relation algebras** and **not containing the unit** e had a **finite basis**.

Theorem

$\mathbf{A} = (\mathbf{A}_0, \circ, ^\smile)$ *is a subreduct of a relation algebra if and only if* $(\mathbf{A}_0, \circ, \triangleright, \triangleleft)$ *with* $x \triangleright y = x^\smile \circ y$ *and* $x \triangleleft y = x \circ y^\smile$ *is an associative r-algebra and* $(x \circ y)^\smile = y^\smile \circ x^\smile$, $(x + y)^\smile = x^\smile + y^\smile$, $x^{\smile\smile} = x$ *and* $x \circ (y/y)(y^-/y^-)$.

In contrast, the following result is also proved:

Theorem

The variety of all r-algebras that can be embedded in r-algebras with a right unit is not finitely based.

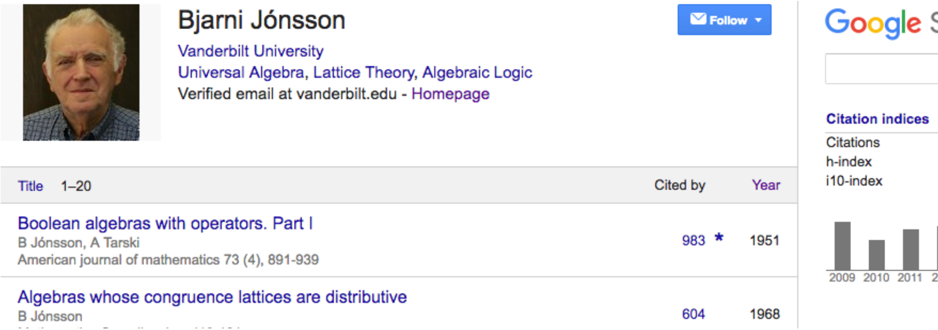
Bounded distributive lattice expansions

85. M. Gehrke, B. Jónsson, Bounded distributive lattices with operators, *Math. Japon.* **40** (2), 207–215, 1994
89. M. Gehrke, B. Jónsson, Monotone bounded distributive lattice expansions, *Math. Japon.* **52** (2), 197–213, 2000
90. M. Gehrke, B. Jónsson, Bounded distributive lattice expansions, *Math. Scand.* **94** (1), 13–45, 2004

These seminal papers **greatly extended** the results of Jónsson and Tarski 1951 to **distributive lattices with arbitrary operations**. Papers by Gehrke and Harding in 2001, and Dunn, Gehrke, and Palmigiano 2005 generalized these results even further to arbitrary **bounded lattices with operators**.

Together with Bjarni's earlier work on relation algebras and residuated Boolean algebras, these results greatly influenced my joint research with **Roger Maddux** on *relation algebras* [1995] and *sequential algebras* [1997] with **Constantine Tsinakis** on the structure of *residuated lattices* [2002], with **Nick Galatos** on *relation algebras as expanded FL-algebras* [2012] and *residuated frames with applications to decidability* [2013], and with **Drew Moshier** on *topological duality and lattice expansions* [2014].

Links to articles that cite Bjarni's work



Clearly Bjarni's research in **universal algebra**, **lattice theory**, **model theory** and **algebraic logic** will continue to have great impact.

Steven Monk, now professor emeritus at the University of Washington, recalled advice that he received from Bjarni Jónsson regarding teaching:
"**Adventure is not in the guidebook and beauty is not on the map. The best one can hope for is to be able to persuade some people to do some traveling on their own.**"

Thank You, Bjarni!

The photo is from the Jónsson Symposium, Laugarvatn, Iceland, July 1990