Nonassociative right hoops

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Overview

- Right residuated posets
- Nonassociative right hoops
- Equational axioms
- Independence of axioms
- Unital nonassociative right hoops
- Characterizing unital congruences

Introduction

A ***residuated magma*** is a partially ordered algebra $(A, \leq, \cdot, /, \backslash)$ such that

- $(A, \leq)$ is a poset,
- $\cdot$ is a binary operation and
- $/, \backslash$ are the right and left residuals of $\cdot$, i.e., the residuation property

$$x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z$$

holds for all $x, y, z \in A$.

As usual, we abbreviate $x \cdot y$ by xy and adopt the convention that $\cdot$ binds stronger than $/, \backslash$

A ***right-residuated magma*** is of the form $(A, \leq, \cdot, /)$ such that $(A, \leq)$ is a poset and

$$xy \leq z \iff x \leq z/y$$

Introduction

Define the term $x \sqcap y = (x/y)y$ and consider the following two varieties:

- A ***right quasigroup*** is an algebra $(A, \cdot, /)$ satisfying the identities

  $$x \sqcap y = x = (xy)/y.$$

  Right quasigroups are precisely those right-residuated magmas for which the partial order $\leq$ is the equality relation.

- A ***right hoop*** is an algebra $(A, \cdot, /)$ satisfying the identities

  $$x \sqcap y = y \sqcap x,\ (x/x)y = y \text{ and } x/(yz) = (x/z)/y.$$

  Then it turns out that x/x is a constant (denoted by 1),
  the operation $\cdot$ is associative,
  the operation $\sqcap$ is a semilattice operation,
  1 is the top element with respect to the semilattice order $\leq$, and
  $/$ is the right residual of $\cdot$ with respect to $\leq$.

Small right quasigroups

$(A, \cdot, /)$ is a right quasigroup **iff** $r_a(x) = xa$ and $q_a(x) = x/a$ are **inverses**
i.e., the columns of the operation table of $\cdot$ are **permutations**

For $A = \{0, 1\}$ with discrete order:

$\cdot$	0	1
0	0	0
1	1	1

$\cdot$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	1	0
1	0	1

$\cdot$	0	1
0	1	1
1	0	0

,**three** nonisomorphic

$|A| = 3, \Rightarrow$ 6 **permutations**, 3 **columns**, so $6^3 = 216$, only 44 up to iso

$|A| = 4, \Rightarrow$ $24^4 = 331776$ quasigroups, only 14022 up to iso

Small right residuated magmas

$(A, \leq, \cdot, /)$ is a right residuated magma **iff** $r_a(x) = xa$ and $q_a(x) = x/a$ are a **unary residuated pair,** hence order preserving and $0a = 0$

For $A = \mathbf{2} = \{0 < 1\}$

$$\begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 0 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 1 \end{array}$$

, so total 7 noniso

For $|A| = 3$ there are 299 right residuated magmas up to isomorphism

$\lvert A\rvert =$	1	2	3	4	5	6
Right res. magmas	1	7	299			
Quasigroups	1	3	44	14022		
Right hoops	1	1	2	8	24	91

2-elt right hoop = BA reduct, 3-elt right hoops = Gödel alg and MV-alg

Introduction

Right hoops were introduced by **Bosbach** [1969,1970] under the name "left complementary semigroups"

Büchi and Owens [1975] studied the case where $\cdot$ is commutative, referring to these structures as "hoops".

The partial order is **definable** in both cases, which motivates the next definition.

Nonassociative right hoops

A ***nonassociative right hoop*** $(A, \leq, \cdot, /)$, or ***narhoop*** for short, is a right-residuated magma such that for all $x, y \in A$

- (N) $\quad x \leq y \quad \iff \quad x \sqcap y = x = y \sqcap x\,.$

In any right-residuated magma $(x/y)y \leq x$ or equivalently $x \sqcap y \leq x$ holds for all x, y,
hence in a narhoop (N) implies that the identity $(x \sqcap y) \sqcap x = x \sqcap y$ holds.

This provides an alternative definition for narhoops:
they are right-residuated magmas that satisfy the identity

- (N1) $\quad (x \sqcap y) \sqcap x = x \sqcap y$ and
- (N') $\quad x \leq y \quad \iff \quad x = y \sqcap x$

since in the presence of (N1), if $x = y \sqcap x$ then multiplying by y on the right we have $x \sqcap y = (y \sqcap x) \sqcap y = y \sqcap x = x$.

Nonassociative (left) hoops

A ***nonassociative left hoop*** or ***nalhoop*** $(A, \leq, \cdot, \backslash)$ is defined dually

A ***nonassociative hoop*** or ***nahoop*** is both a narhoop and a nalhoop

We consider only na**r**hoops in this talk.

The two motivating varieties fit into this framework as follows:

- A narhoop $(A, \leq, \cdot, /)$ is a right quasigroup **if and only if** $\leq$ is the equality relation.
- A narhoop $(A, \leq, \cdot, /)$ is a right hoop **if and only if** $x/yz = (x/z)/y$ and the quasiequation $x \sqcap y = x \Rightarrow x \leq y$ holds.

$\lvert A \rvert =$	1	2	3	4	5	6
Right res. magmas	1	7	299			
Narhoops	**1**	**4**	**52**	**14607**		
Quasigroups	1	3	44	14022		
Right hoops	1	1	2	8	24	91

Main result

Narhoops form a finitely based variety of algebras.

We assume that x/y binds stronger than $x \sqcap y = (x/y)y$.

Theorem

Let $(A, \leq, \cdot, /)$ *be a narhoop. Then the following identities hold:*

(N1) $(x \sqcap y) \sqcap x = x \sqcap y$

(N2) $xy/y \sqcap x = x$

(N3) $xz \sqcap (x \sqcap y)z = (x \sqcap y)z$

(N4) $(x/z) \sqcap (x \sqcap y)/z = (x \sqcap y)/z.$

Conversely, let $(A, \cdot, /)$ *be an algebra with two binary operations satisfying (N1)–(N4), and define* $x \leq y \iff x = y \sqcap x$*. Then the identities*

(N5) $x \sqcap xy/y = x$

(N6) $(x \sqcap y)/y = x/y$

(N7) $(x \sqcap y) \sqcap y = x \sqcap y$

hold and $(A, \leq, \cdot, /)$ ***is a narhoop****.*

Proof.

Assume $(A, \leq, \cdot, /)$ is a narhoop.
As noted before, the identity (N1: $(x \sqcap y) \sqcap x = x \sqcap y$) holds in narhoops

Right-residuated magmas also satisfy $x \leq xy/y$
hence (N2) $xy/y \sqcap x = x$ follows from (N': $x \leq y \iff x = y \sqcap x$)

Having a right residual implies that right-multiplication is order preserving so $(x \sqcap y)z \leq xz$ holds in all narhoops, which produces (N3).

Similarly the right residual is order preserving in the first argument, hence $(x \sqcap y)/z \leq x/z$ holds, and now (N4) follows from (N').

Proof continued.

Conversely, suppose $(A, \cdot, /)$ satisfies (N1)–(N4), and $\leq$ is defined by (N')

From (N2: $xy/y \sqcap x = x$), (N1: $(x \sqcap y) \sqcap x = x \sqcap y$) and (N2) again, we get (N5):

$$x \sqcap (xy/y) = (xy/y \sqcap x) \sqcap (xy/y) = (xy/y) \sqcap x = x$$

Replace x in (N5) by x/y to get $x/y \sqcap (x \sqcap y)/y = x/y$
then use (N4: $(x/z) \sqcap (x \sqcap y)/z = (x \sqcap y)/z$) to get N6: $(x \sqcap y)/y = x/y$

To prove (N7: $(x \sqcap y) \sqcap y = x \sqcap y$) multiply (N6) on the right by y

Proof continued.

Now reflexivity of $\leq$ follows from (N5) and (N1):
$x \sqcap x = (x \sqcap xy/y) \sqcap x = x \sqcap (xy/y) = x$.

For antisymmetry, if $x \leq y$ and $y \leq x$, then $x \sqcap y = x = y \sqcap x$ and $y = x \sqcap y$, hence $x = y$.

Transitivity: Suppose $x \leq y$ and $y \leq z$ so that $x \sqcap y = x = y \sqcap x$ and $y \sqcap z = y = z \sqcap y$. First, note that

$$z/x \sqcap y/x = z/x \sqcap (z \sqcap y)/x = (z \sqcap y)/x = y/x$$

using (N4) in the second equality.

Proof continued.

Now we compute

$$\begin{aligned} z \sqcap x &= (z \sqcap x) \sqcap x && \text{by N6: } (x \sqcap y)/y = x/y \\ &= (z \sqcap x) \sqcap (y \sqcap x) = (z/x)x \sqcap (y/x)x \\ &= (z/x)x \sqcap (z/x \sqcap y/x)x && \text{since } z/x \sqcap y/x = y/x \\ &= (z/x \sqcap y/x)x && \text{by (N3)} \\ &= (z/x \sqcap (z \sqcap y)/x)x = ((z \sqcap y)/x)x && \text{by (N4)} \\ &= (z \sqcap y) \sqcap x = y \sqcap x = x. \end{aligned}$$

From $x = z \sqcap x$ we deduce $x \sqcap z = (z \sqcap x) \sqcap z = z \sqcap x$ by (N1), hence $x \leq z$.

Proof continued.

Finally, we prove $/$ is the right residual of $\cdot$ with respect to $\leq$

The right residuation property is equivalent to $(x/y)y \leq x \leq xy/y$ and $x \leq y$ implies $xz \leq yz$ and $x/z \leq y/z$

Note that (N2) and (N') show $x \leq xy/y$.

If $x \leq y$, then (N3) gives

$$yz \sqcap xz = yz \sqcap (y \sqcap x)z = (y \sqcap x)z = xz,$$

and so (N') implies $xz \leq yz$.

By the same argument, (N4) gives $x/z \leq y/z$.

Proof continued.

To prove $(x/y)y \leq x$, or equivalently $x \sqcap y \leq x$,

substitute x/x for x, x for y, and $(x \sqcap y)/x$ for z in (N3) to get

$$(x/x)x \sqcap (x/x \sqcap (x \sqcap y)/x)x = (x/x \sqcap (x \sqcap y)/x)x$$

Using (N4) this simplifies to $(x \sqcap x) \sqcap ((x \sqcap y)/x)x = ((x \sqcap y)/x)x$

so by (N1), (N') and reflexivity we have $x \sqcap y \leq x \sqcap x = x$

Independence of axioms

The equational basis (N1)–(N4) for narhoops is independent as can be seen from algebras $A_i = \{0, 1\}$ ($i = 1, 2, 3, 4$) that each satisfy the axioms except for (Ni).

- In A_1, $\cdot$ is ordinary multiplication and $x/y = y$.
- In A_2, $x \cdot y = x$ and $x/y = 1$.
- In A_3, $x \cdot y$ is addition modulo 2 and $x/y = 0$ except that $1/0 = 1$.
- In A_4, $x \cdot y$ is the max operation and x/y is addition modulo 2.

In general, neither $\cdot$ nor the term operation $\sqcap$ of a narhoop is associative.

However $\sqcap$ is associative both in right quasigroups and in right hoops.

In right quasigroups, this follows from the identity $x \sqcap y = x$.

In right hoops, $\sqcap$ turns out be a semilattice operation (J. 2017, Lem. 4).

In both cases the reduct $(A, \sqcap)$ is a *left normal band*, i.e., an idempotent semigroup satisfying the identity $x \sqcap y \sqcap z = x \sqcap z \sqcap y$.

Narhoops with associative $\sqcap$

If $(A, \cdot, /)$ is a narhoop and $B \subseteq A$ is closed under $\sqcap$, then B inherits the order $\leq$ from A.

Theorem

Let $(A, \cdot, /)$ be a narhoop and let $B \subseteq A$ be closed under $\sqcap$. The following are equivalent.

1. $(B, \sqcap)$ *is a left normal band;*
2. $(B, \sqcap)$ *is a semigroup;*
3. *For all* $x, y \in B$, $x \sqcap (y \sqcap x) = x \sqcap y$ *and* $(x \sqcap (y \sqcap z)) \sqcap z = x \sqcap (y \sqcap z)$.

Narhoops with commutative $\sqcap$

$\sqcap$-reducts of right hoops are semilattices

A nonassociative generalization of right hoops is the following variety of narhoops:

Theorem

Let $(A, \cdot, /)$ be a narhoop and let $B \subseteq A$ be closed under $\sqcap$. The following are equivalent.

1. *$(B, \sqcap)$ is commutative;*
2. *For all $x, y \in B$, $x \sqcap (y \sqcap x) = y \sqcap x$.*

When these equivalent conditions hold, $(B, \sqcap)$ is a semilattice.

Principal ideals of narhoops

In a left normal band, the identity $x \sqcap y \sqcap z = x \sqcap z \sqcap y$ expresses the fact that every downset $(a] = \{x \in A \mid x \leq a\} = \{a \sqcap x \mid x \in A\}$ is a subsemilattice.

The same role is played by $(x \sqcap y) \sqcap z = (x \sqcap z) \sqcap y$ in narhoops.

Theorem

Let $(A, \cdot, /)$ be a narhoop that satisfies $(x \sqcap (y \sqcap z)) \sqcap z = x \sqcap (y \sqcap z)$, and fix $a \in A$. Then the downset $(a]$ is closed under $\sqcap$ and is a semilattice.

Narhoops with a left identity element

Lemma

Let $(A, \leq, \cdot, /)$ *be a right-residuated magma such that* $x \leq y \iff x = y \sqcap x$ *holds for all* $x, y \in A$. *Then*

1. x/x *is a maximal element for all* $x \in A$,
2. *the identity* $(x/x)y/y = x/x$ *holds in* A, *and*
3. *if* A *has a top element then the term* x/x *is this top element.*

Unital narhoops

Lemma

Let $(A, \cdot, /)$ be a narhoop. The following are equivalent.

1. $x/x = y/y$ *for all* $x, y \in A$;
2. $(x/x)y = y$ *for all* $x, y \in A$;
3. *There exists* $e \in A$ *such that* $ey = y$ *for all* $y \in A$.

When these conditions hold, the element $1 = x/x$ *is the maximum left identity element in* $(A, \leq)$.

A narhoop $(A, \cdot, /)$ is ***unital*** if the equivalent conditions of this lemma hold.

In this case we denote by $1 = x/x$ the distinguished left identity element.

Note that the lemma does not claim that 1 is the unique left identity element, even when $\sqcap$ is commutative.

Theorem

If A is finite and unital, then 1 is the unique left identity element.

In a unital narhoop A the partial order $\leq$ can be **characterized** in terms of 1 and $/$:

$$x \leq y \iff x/y = 1$$

for all $x, y \in A$.

Furthermore, the left identity 1 can also be used to characterize the commutativity of $\sqcap$.

Theorem

Let $(A, \cdot, /, 1)$ be a unital narhoop. Then:

1. *The left unit 1 is* ***the top element*** *of $(A, \leq)$ if and only if $\sqcap$ is commutative.*
2. *The downset $(1]$ is a subnarhoop and $((1], \sqcap)$ is a semilattice.*

A congruence θ on a narhoop $(A, \cdot, /)$ is said to be ***unital*** if the factor narhoop A/θ is unital.

In other words, θ is unital if and only if $x/x \;\theta\; y/y$ for all $x, y \in A$.

If A itself is unital then every congruence on A is unital.

For a unital congruence on an arbitrary narhoop, set

$$\begin{aligned} N_\theta &= \{x \in A \mid x \;\theta\; y/y \text{ for some } y \in A\} \\ &= \{x \in A \mid x \;\theta\; y/y \text{ for all } y \in A\}, \end{aligned}$$

where the second equality follows since θ is unital.

Analogous to the relationship between congruences and normal subgroups in group theory, we claim that θ is determined by the congruence class N_θ.

To state the result concisely, we introduce six families of mappings on a narhoop $(A, \cdot, /)$.

For each $x, y \in A$, $i = 1, \ldots, 6$, define $\phi_{i,x,y} : A \to A$ by

$$\begin{aligned} \phi_{1,x,y}(z) &= (zx \cdot y)/xy\,, & \phi_{2,x,y}(z) &= (zx/y)/(x/y)\,, \\ \phi_{3,x,y}(z) &= (x \cdot zy)/xy\,, & \phi_{4,x,y}(z) &= (x/zy)/(x/y)\,, \\ \phi_{5,x,y}(z) &= xy/(x \cdot zy)\,, & \phi_{6,x,y}(z) &= (x/y)/(x/zy)\,. \end{aligned}$$

Keeping analogies with group theory in mind, let $\mathrm{Inn}(A)$ denote the transformation semigroup on A generated by these six families of mappings.

Theorem

Let θ be a unital congruence on a narhoop $(A, \cdot, /)$. Then:

1. *N_θ is a subnarhoop of A;*
2. *For all $x, y \in A$, if $x \leq y$ and $x \in N_\theta$, then $y \in N_\theta$;*
3. *N_θ is invariant under $\mathrm{Inn}(A)$.*

Let $(A, \cdot, /)$ be a narhoop. A nonempty subset N of A is said to be a *normal subnarhoop* of A, denoted $N \trianglelefteq A$, if the following hold:

1. N is a subnarhoop of A;
2. For all $x, y \in A$, if $x \leq y$ and $x \in N$, then $y \in N$;
3. N is invariant under $\mathrm{Inn}(A)$.

Theorem

Let $(A, \cdot, /)$ be a narhoop and assume $N \trianglelefteq A$ is nonempty. Define θ_N on A by $x \; \theta_N \; y$ if and only if $x/y, y/x \in N$. Then θ_N is a unital congruence and $N_{\theta_N} = N$.

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