From Residuated Lattices to Boolean Algebras with Operators

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Outline

- **Partially ordered sets** and **lattices**
- **Residuated lattices** and **substructural logics**
- **Heyting algebras** and **Boolean algebras**
- **Boolean algebras with operators** (BAOs)
- BAOs with a **monoid operator** are **not equationally decidable**
- **Heyting algebras with operators** (HAOs)
- **Generalized bunched implication algebras** (GBI-algebras)
- **Residuated lattices**, **GBI-algebras** and several of their subclasses are **equationally decidable**
- Some **algorithms** for computing with **finite algebras**

Partially ordered sets

Let P be a set and $\leq$ a binary relation on P

This means $\leq$ is a subset of $P \times P = \{(a, b) : a, b \in P\}$

Write $a \leq b$ iff (a, b) is in $\leq$

$(P, \leq)$ is a **partially ordered set** (or **poset**) if

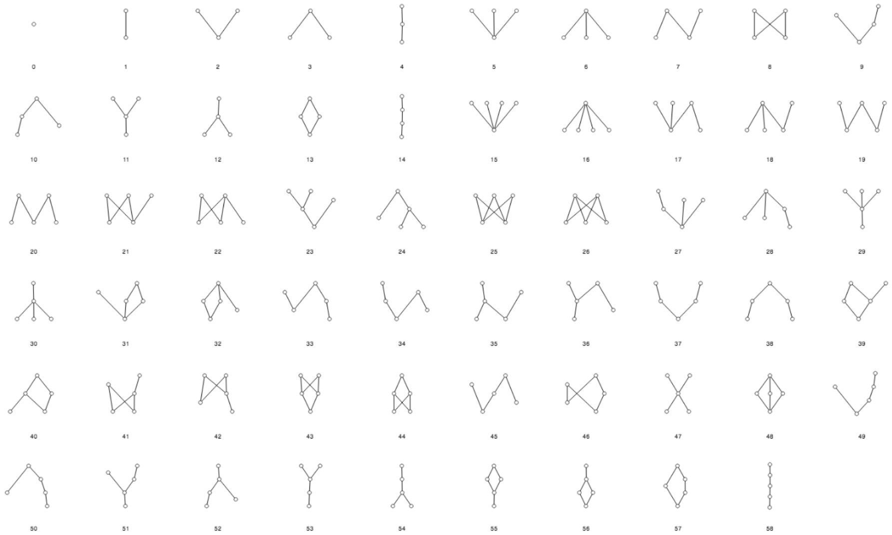
1. $\leq$ is reflexive: $x \leq x$
2. $\leq$ is antisymmetric: $x \leq y$ and $y \leq x$ imply $x = y$
3. $\leq$ is transitive: $x \leq y$ and $y \leq z$ imply $x \leq z$ for all $x, y, z \in P$

Alternative logical notation (from proof theory):

$$\frac{}{x \leq x}\ (\mathit{refl}) \qquad \frac{x \leq y \quad y \leq x}{x = y}\ (=) \qquad \frac{x \leq y \quad y \leq z}{x \leq z}\ (\mathit{cut})$$

Example: $(P, \leq) = (\{0, 1, 2\}, \{(0, 0), (1, 1), (2, 2), (0, 1), (0, 2)\})$

The 59 connected posets of size ≤ 5



Semilattices

Let $(S, \leq)$ be a poset and $\wedge$ a binary operation on S

This means $\wedge$ is a function from $S \times S$ to S

Write $a \wedge b$ for the value of $\wedge(a, b)$

$(S, \wedge, \leq)$ is a **semilattice** if (logical version)

$$\frac{x \leq z}{x \wedge y \leq z}\ (\wedge_{L1}) \qquad \frac{y \leq z}{x \wedge y \leq z}\ (\wedge_{L2}) \qquad \frac{x \leq y \quad x \leq z}{x \leq y \wedge z}\ (\wedge_R)$$

or equivalently (algebraic version)

1. $\wedge$ is associative: $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
2. $\wedge$ is commutative: $x \wedge y = y \wedge x$
3. $\wedge$ is idempotent: $x \wedge x = x$
4. $x \leq y$ iff $x \wedge y = x$ for all $x, y, z \in S$

Example: $(S, \wedge, \leq) = (\{0, 1, 2\}, \{0 \wedge 1 = 0 = 0 \wedge 2 = 1 \wedge 2\}, \leq)$

Lattices

Let $(L, \wedge, \leq)$ be a semilattice and $\vee$ another binary operation on L

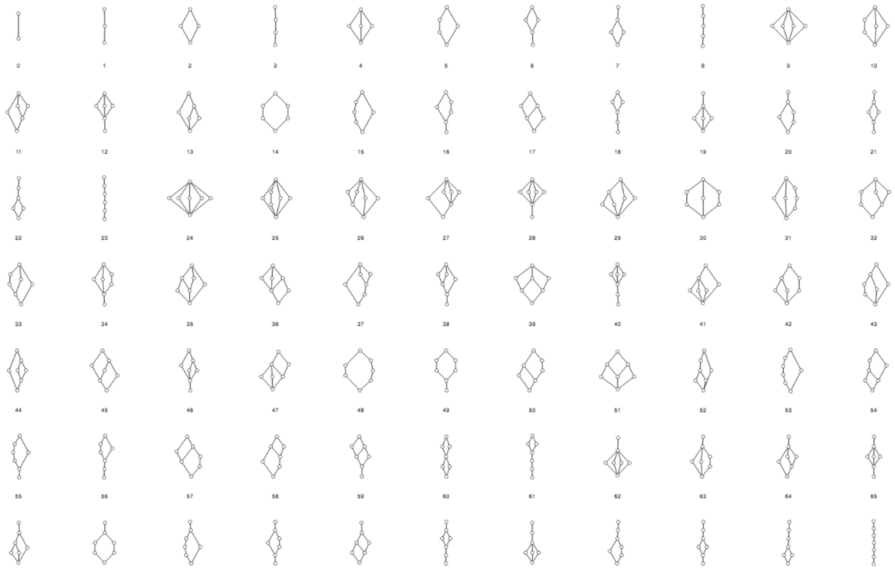
$(L, \wedge, \vee, \leq)$ is a **lattice** if (logical version)

$$\frac{x \leq z \quad y \leq z}{x \vee y \leq z}\ (\vee_L) \qquad \frac{x \leq y}{x \leq y \vee z}\ (\vee_{R1}) \qquad \frac{x \leq z}{x \leq y \vee z}\ (\vee_{R2})$$

or equivalently (algebraic version)

1. $\vee$ is an associative, commutative operation and
2. $\vee$ is absorbtive: $x \vee (x \wedge y) = x$
3. $\wedge$ is absorbtive: $x \wedge (x \vee y) = x$
4. $x \leq y$ iff $x \vee y = y$

The 77 nontrivial lattices of size ≤ 7



Residuated lattices

Let $(A, \wedge, \vee, \leq)$ be a lattice, $\cdot, \backslash, /$ binary operations on A, and $1 \in A$

$(A, \wedge, \vee, \cdot, 1, \backslash, /, \leq)$ is a **residuated lattice** if (algebraic version)

1. $\cdot$ is associative: $(xy)z = x(yz)$
2. 1 is an identity element: $1x = x = x1$ (so $(A, \cdot, 1)$ is a **monoid**)
3. $\backslash, /$ are the left and right residual of $\cdot$:

$$xy \leq z \quad \iff \quad y \leq x\backslash z \quad \iff \quad x \leq z/y \quad \text{for all } x, y, z \in A$$

or equivalently (logical version; double bar means $\iff$)

$$\frac{x(yz) \leq w}{\overline{(xy)z \leq w}} \qquad 1x = x = x1 \qquad \frac{xy \leq z}{y \leq x\backslash z} \qquad \frac{xy \leq z}{x \leq z/y}$$

$$\frac{x \leq z \quad y \leq w}{xy \leq zw} \qquad \frac{x \leq y \quad uzv \leq w}{ux(y\backslash z)v \leq w} \qquad \frac{x \leq y \quad uzv \leq w}{u(z/y)xv \leq w}$$

Selfcontained equational definition of residuated lattices

An algebra $(A, \wedge, \vee, \cdot, 1, \backslash, /)$ is a **residuated lattice** if the following **equations** hold for all $x, y, z \in A$:

$$
\begin{array}{ccc}
(x \vee y) \vee z = x \vee (y \vee z) & (xy)z = x(yz) & x(x\backslash z \wedge y) \vee z = z \\
(x \wedge y) \wedge z = x \wedge (y \wedge z) & x1 = x = x1 & x\backslash(xz \vee y) \wedge z = z \\
x \vee y = y \vee x & x \vee (x \wedge y) = x & (y \wedge z/x)x \vee z = z \\
x \wedge y = y \wedge x & x \wedge (x \vee y) = x & (y \vee zx)/x \wedge z = z
\end{array}
$$

Therefore residuated lattices form an **equational class** (hence closed under direct products, subalgebras and homomorphic images)

As before, define $x \leq y$ if and only if $x \wedge y = x$

Birkhoff's equational logic is used to derive other equations

Counterexamples (specific residuated lattices) are used to show that certain formulas do **not** hold in all residuated lattices

Residuated lattices and substructural logics

Residuated lattices correspond to **propositional substructural logic**

x, y, z are propositions, $\wedge, \vee, \cdot, \backslash, /$ are logical connectives

The operation $\cdot$ is a **noncommutative linear** conjunction, called **fusion**

$x \implies y$ is **true** iff $x \le y$ iff $1 \le x\backslash y$

Residuated lattices generalize many algebras related to **logic**, e. g. **Boolean algebras**, **Heyting algebras**, **MV-algebras**, **Gödel algebras**, **Product algebras**, Hajek's **basic logic algebras**, **linear logic algebras**, **relation algebras**, **lattice-ordered (pre)groups**, ...

Residuated lattices were originally defined and studied by **Ward and Dilworth** [1939]

The set of **ideals of a ring** forms a residuated lattice

Hiroakira Ono

(California, September 2006)

[1985] *Logics without the contraction rule*

(with Y. Komori)

Provides a **framework** for studying many substructural logics, relating sequent calculi with semantics

The name **substructural logics** was suggested by K. Dozen, October 1990

[2007] *Residuated Lattices: An algebraic glimpse at substructural logics* (with Galatos, J., Kowalski)

Heyting algebras

A residuated lattice $(A, \wedge, \vee, \cdot, 1, \backslash, /)$ is a **Heyting algebra** if

$x \leq 1$, $xy = x \wedge y$, and A has a **bottom element** $0 \leq x$

In this case, $z \leq x\backslash y \iff x \wedge z \leq y \iff z \wedge x \leq y \iff z \leq y/x$

So $x\backslash y = y/x$ and this value is denoted $x \to y$

Therefore a Heyting algebra is of the form $(A, \wedge, \vee, \to, 0, 1)$

In any residuated lattice $x(y \vee z) \leq w \iff y \vee z \leq x\backslash w \iff y \leq x\backslash w$ and $z \leq x\backslash w \iff xy \leq w$ and $xz \leq w \iff xy \vee xz \leq w$

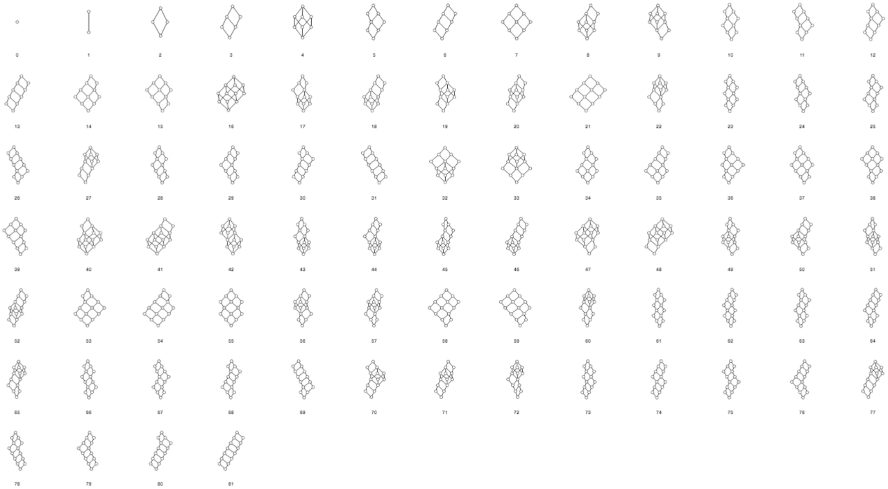
Therefore $x(y \vee z) = xy \vee xz$

In a Heyting algebra this means $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

So any Heyting algebra is a distributive lattice

Also any finite distributive lattice is a Heyting algebra

82 vertically indecomposable Heyting algebras of size ≤ 14



Boolean algebras

In a Heyting algebra we define $\neg x = x \to 0$

This unary operation satisfies $x \wedge y \leq 0 \iff y \leq x \to 0 \iff y \leq \neg x$

So $\neg x$ is the largest element y such that $x \wedge y = 0$

$\neg x$ is called the **pseudocomplement** of x

A Heyting algebra $(A, \wedge, \vee, \to, 0, 1)$ is a **Boolean algebra** if

$$\neg\neg x = x \quad \text{for all } x \in A$$

In this case $x \to y = \neg x \vee y$, hence $\neg x \vee x = x \to x = 1$

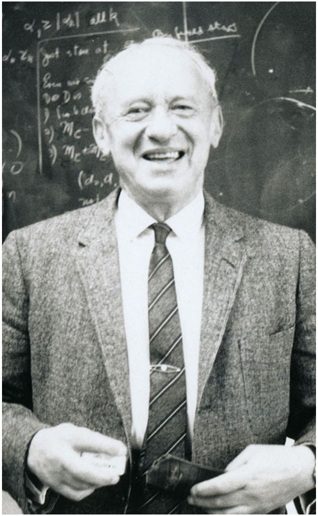
Therefore $\neg x$ is a **complement** in any Boolean algebra $(A, \wedge, \vee, \neg, 0, 1)$

Example: $B_1 = (\{0, 1\}, \min, \max, 1-x, 0, 1)$ is the 2-element BA

Every **finite** Boolean algebra is of the form $(B_1)^n$ for some finite n

Alternatively, every finite BA is of the form $\mathcal{P}(X)$ for a set X with n

Algebraic logic



Alfred Tarski

(May 1967, visiting at U. of Michigan)

According to the MacTutor Archive, **Tarski** is recognised as one of the four greatest logicians of all time, the other three being **Aristotle**, **Frege**, and **Gödel**

Of these **Tarski** was the most prolific as a logician

His collected works, excluding the 20 books, runs to 2500 pages

Boolean algebras with operators

Bjarni Jónsson

(AMS-MAA meeting in Madison, WI 1968)

Boolean Algebras with operators, Part I and Part II [1951/52] with **Alfred Tarski**

One of the cornerstones of algebraic logic

Constructs **canonical extensions** and provides **semantics** for multi-modal logics

Gives representation for **abstract relation algebras** by **atom structures**

Boolean algebras with operators

Let $\tau = \{f_i : i \in I\}$ be a set of operation symbols, each with a fixed finite arity

$\mathbf{BAO}_\tau$ is the class of algebras $(A, \vee, \wedge, \neg, \bot, \top, f_i\,(i \in I))$ such that $(A, \vee, \wedge, \neg, \bot, \top)$ is a **Boolean algebra** and the f_i are **operators** on A

i.e., $f_i(\ldots, x \vee y, \ldots) = f_i(\ldots, x, \ldots) \vee f_i(\ldots, y, \ldots)$

and $f_i(\ldots, \bot, \ldots) = \bot$ for all $i \in I$ (so the f_i are **strict**)

BAOs are the **algebraic semantics** of classical **multimodal logics**

Main result [Jonsson-Tarski 1952]: every BAO $\mathbf{A}$ can be embedded in its **canonical extension** $\mathbf{A}^\sigma$, a **complete and atomic** Boolean algebra with operators

The **set of atoms** of this Boolean algebra is the **Kripke frame** of the multimodal logic

Example: Residuated Boolean monoids

A **residuated Boolean monoid** is an algebra $(A, \vee, \wedge, \neg, \bot, \top, \cdot, 1, \triangleright, \triangleleft)$ such that $(A, \vee, \wedge, \neg, \bot, \top)$ is a **Boolean algebra**, $(A, \cdot, 1)$ is a **monoid** and for all $x, y, z \in A$

$$(x \cdot y) \wedge z = \bot \iff (x \triangleright z) \wedge y = \bot \iff (z \triangleleft y) \wedge x = \bot$$

Rewrite this as

$$x \cdot y \leq z \iff y \leq \neg(x \triangleright \neg z) \iff x \leq \neg(\neg z \triangleleft y)$$

Define $x \backslash z = \neg(x \triangleright \neg z)$ and $z / y = \neg(\neg z \triangleleft y)$, to see that residuated Boolean monoids are term-equivalent to Boolean residuated lattices (i.e., the lattice structure is Boolean)

Theorem

[Jónsson, Tsinakis 1992] ***Relation algebras*** *are a subvariety of residuated Boolean monoids*

$\implies$ **Relation algebras** are (term-equivalent to) $\subseteq$ **Boolean residuated lattices**

Boolean + associative operator ⇒ undeciable

Theorem

[Tarski 1941] The class of ***representable relation algebras*** *has an* ***undecidable equational theory****, and the same holds for the class of* ***(abstract) relation algebras***

Theorem

[Andreka, Kurucz, Nemeti, Sain, Simon 95, 96] The equational theories of ***residuated Boolean monoids*** *and* ***commutative residuated Boolean monoids****, as well as a large interval of other classes, are* ***undecidable***

Lattices and Heyting algebras with operators

Posets $\supset$ Semilattices $\supset$ Lattices $\supset$ Heyting algebras $\supset$ Boolean algebras

Then we added operators (operations that distribute over $\vee$)

$\Longrightarrow$ Lattices with operators (LO) and

Heyting algebras with operators (HAO)

$$\begin{array}{ccc}
\mathbf{LO} & \supset & \text{Residuated lattices} \\
| & & | \\
\mathbf{HAO} & \supset & \text{Generalized bunched implication algebras} \\
| & & | \\
\mathbf{BAO} & \supset & \text{Residuated Boolean monoids}
\end{array}$$

HAO example: Generalized bunched implication algebras

Recall that a **Heyting algebra** is a residuated lattice with $0 = \perp$ as bottom element and $xy = x \wedge y$

In this case we write $x \to y$ instead of $x\backslash y \quad (= y/x)$

Also define $\neg x = x \to \perp$ and $\top = \neg\perp$

A **generalized bunched implication algebra** or **GBI-algebra** is an algebra $(A, \vee, \wedge, \to, \perp, \cdot, 1, \backslash, /)$ where $(A, \vee, \wedge, \to, \perp)$ is a **Heyting algebra**, and $(A, \vee, \wedge, \cdot, 1, \backslash, /)$ is a **residuated lattice**

Theorem

[J. & Galatos 2015] The equational theory of GBI-algebras is ***decidable***

BI-algebras are commutative GBI-algebras

Applications in computer science; basis of **separation logic**

A glimpse of algebraic proof theory

Gentzen [1936] defined **sequent calculi**, including **LK** (for classical logic) and **LJ** (for intuistionistic logic)

For **proof search** and **proof normalization**, he proved that the **cut rule** can be **omitted** without affecting provability

Example: A a **sequent calculus** for residuated lattices

Let **RL** be the **equational theory** of **residuated lattices**

Let $T = Fm_{\vee,\wedge,\cdot,1,\backslash,/}(x_1, x_2, \ldots)$, $\quad W = F_{Mon(\circ,\varepsilon)}(T)$, $\quad W' = U \times T$

where $U = \{u \in F_{Mon(\circ,\varepsilon)}(T \cup \{x_0\}) : u$ contains **exactly one** $x_0\}$

The Gentzen system GL

A **Horn formula** $\varphi_1 \& \cdots \& \varphi_n \to \psi$ is written $\frac{\varphi_1 \ \cdots \ \varphi_n}{\psi}$

Let $a, b, c \in T$, $\quad s, t \in W \quad$ and $\quad u \in U$

GL: $\quad \dfrac{}{a \Rightarrow a} \qquad \dfrac{t \Rightarrow a}{t \Rightarrow a \vee b} \qquad \dfrac{t \Rightarrow b}{t \Rightarrow a \vee b} \qquad \dfrac{u(a) \Rightarrow c \quad u(b) \Rightarrow c}{u(a \vee b) \Rightarrow c}$

$\dfrac{t \Rightarrow a \quad u(a) \Rightarrow b}{u(t) \Rightarrow b}(\text{cut}) \qquad \dfrac{u(a) \Rightarrow c}{u(a \wedge b) \Rightarrow c} \qquad \dfrac{u(b) \Rightarrow c}{u(a \wedge b) \Rightarrow c} \qquad \dfrac{t \Rightarrow a \quad t \Rightarrow b}{t \Rightarrow a \wedge b}$

$\dfrac{u(a \circ b) \Rightarrow c}{u(a \cdot b) \Rightarrow c} \qquad \dfrac{s \Rightarrow a \quad t \Rightarrow b}{s \circ t \Rightarrow a \cdot b} \qquad \dfrac{}{\varepsilon \Rightarrow 1} \qquad \dfrac{u(\varepsilon) \Rightarrow a}{u(1) \Rightarrow a}$

$\dfrac{a \cdot t \Rightarrow b}{t \Rightarrow a \backslash b} \qquad \dfrac{t \Rightarrow a \quad u(b) \Rightarrow c}{u(t \circ (a \backslash b)) \Rightarrow c} \qquad \dfrac{t \cdot b \Rightarrow a}{t \Rightarrow a / b} \qquad \dfrac{t \Rightarrow b \quad u(a) \Rightarrow c}{u((a/b) \circ t) \Rightarrow c}$

Example of a *cut-free* **RL** proof $\dfrac{\dfrac{\dfrac{\dfrac{z \Rightarrow z \quad x \Rightarrow x}{z \circ (z \backslash x) \Rightarrow x}}{z \circ (z \backslash x \wedge z \backslash y) \Rightarrow x} \quad \dfrac{\dfrac{z \Rightarrow z \quad y \Rightarrow y}{z \circ (z \backslash y) \Rightarrow y}}{z \circ (z \backslash x \wedge z \backslash y) \Rightarrow y}}{z \circ (z \backslash x \wedge z \backslash y) \Rightarrow x \wedge y}}{z \backslash x \wedge z \backslash y \Rightarrow z \backslash (x \wedge y)}$

Semantics of sequent calculi: Residuated frames

Let $\mathbf{GL}_{cf}$ be the sequent calculus **GL without** the **cut rule**

Define a binary relation $N \subseteq W \times W'$ by

$$wN(u,a) \iff u(w) \Rightarrow a \quad \text{is provable in } \mathbf{GL}_{cf}$$

Define the **accessibility** relations $R_\circ \subseteq W^3$, $R_{\backslash\!\backslash}, R_{/\!/}$ by

$$R_\circ(v_1, v_2, w) \iff v_1 \circ v_2 = w$$

$$R_{\backslash\!\backslash} = \{((u,a), x, (u(\_\circ x), a)) : u \in U, a \in T, x \in W\}$$

$$R_{/\!/} = \{(x, (u,a), (u(x \circ \_), a)) : u \in U, a \in T, x \in W\}$$

Then $(W, W', N, R_\circ, R_{\backslash\!\backslash}, R_{/\!/})$ is a **residuated frame**

(A **general** residuated frame is $(W, W', N, R_i(i \in I))$)

Algebraic cut-admissibility

Theorem

[Okada, Terui 1999, Galatos, J. 2013]. The following are equivalent:

1. $t \Rightarrow a$ *is provable in* $\boldsymbol{GL}$
2. $t \leq a$ *holds in* $\boldsymbol{RL}$
3. $t \Rightarrow a$ *is provable in* $\boldsymbol{GL}_{cf}$

Proof (outline): (3⇒1) is obvious. (1⇒2) Assume $t \Rightarrow a$ is provable **with cut**. Show that **all sequent rules** hold as quasiequations in **RL** (where $\Rightarrow, \circ$ are **replaced by** $\leq, \cdot$)

(2⇒3) Assume $t \leq a$ holds in **RL** and define an algebra $\mathbf{W}^+ = (C[\mathcal{P}(W)], \cup, \cap, \cdot, 1, \backslash, /)$ using the **closed sets** $C(X)$ of the **polarity** (W, W', N) and

$X \cdot Y = C(\{w : R(v_1, v_2, w) \text{ for some } v_1 \in X, v_2 \in Y\})$

$X\backslash Y = \{w \in W : X \cdot \{w\} \subseteq Y\}$ $\qquad$ $Y/X = \{w \in W : \{w\} \cdot X \subseteq Y\}$.

Proof outline (continued)

Then $\mathbf{W}^+$ is a residuated lattice, hence satisfies $t \le a$

Let $f : T \to \mathbf{W}^+$ be a **homomorphism**

Extend to $\bar{f} : W \to \mathbf{W}^+$, so $t \le a$ implies $\bar{f}(t) \subseteq \bar{f}(a)$

Define $\{b\}^{\triangleleft} = \{w \in W : wN(x_0, b)\}$

Prove by **induction** that $b \in \bar{f}(b) \subseteq \{b\}^{\triangleleft}$ for all $b \in T$

Then $t \in \bar{f}(t) \subseteq \bar{f}(a) \subseteq \{a\}^{\triangleleft}$, hence $tN(x_0, a)$

Therefore $\quad t \Rightarrow a \quad$ holds in $\mathbf{GL}_{cf}$ $\square$

Theorem

The equational theory of residuated lattices is ***decidable****. Moreover,* **RL** *has the* ***finite model property***

[Galatos, J. 2013] The variety of integral RL (i.e., $x \wedge 1 = x$) has the ***finite embedding property****, hence the* ***universal theory is decidable****.*

Expanding this approach to GBI-algebras

A similar approach can be used to prove that the equational theory of GBI-algebras is decidable

Add Gentzen rules for an external connective $\owedge$ corresponding to $\wedge$, and rules for $\to$

Expand the residuated frame with a ternary relation for $\owedge$

Theorem

[Galatos & J. 2015] The equational theory of GBI-algebras is ***decidable****. Moreover, (G)BI-algebras have the* ***finite model property***

Theorem

[Galatos & J. 2015] The variety of integral GBI-algebras (i.e., $x \wedge 1 = x$) has the ***finite embedding property****, hence the* ***universal theory is decidable****.*

How to compute finite residuated lattices

First compute all **lattices** with n elements (up to isomorphism)

[J. and Lawless 2015]: For $n = \mathbf{19}$ there are **1 901 910 625 578**

Then compute all **lattice-ordered monoids** with **zero** ($\perp$) over each lattice

The residuals are **determined** by the monoid

There are **295292 residuated lattices** of size $n = \mathbf{8}$

[Belohlavek and Vychodil 2010]: For **commutative integral** residuated lattices there are **30 653 419** of size $n = \mathbf{12}$

Conclusion

Residuated lattices are an excellent **framework** for investigating and **comparing** algebras related to propositional logics

By adding **operators** many more propositional logics are covered

Between **HAOs** and **BAOs** there is much **uncharted territory**

The success of **bunched implication logic** and **separation logic** in **program verification** provide justification for **more research** in this area

Algebraic, **semantic** and **logical** techniques can often be adapted to HAOs and LOs

Some References

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Thank You