An introduction to residuated lattices

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Some remarks

No background of (residuated) lattices or algebraic logic is assumed.

A handout is provided with some crucial formulas missing.

On the slides the missing formulas appear in ***red***.

Your activity consists of copying these formulas into the correct blank.

Ask questions if you want me to slow down (or stop me from asking questions).

Residuated lattice have many applications in logic, computer science, linguistics,...

These are (unfortunately) not the focus of this brief introduction.

Semigroups and monoids

A **semigroup** $(A, \cdot)$ is a set A with an associative binary operation:
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

oeis.org/A058129: $\begin{cases} 1, 5, 24, 188, 1915, 28634, 1627672, 3684030417, \\ 105978177936292\ (n = 9) \end{cases}$

A **monoid** $(A, \cdot, 1)$ is a semigroup with an identity 1, i.e.:
$x \cdot 1 = x = 1 \cdot x$

oeis.org/A027851: $\{1, 2,\ 7,\ 35,\ 228,\ 2237,\ \ 31559,\ 1668997\ (n = 8)$

A **preordered set** $(A, \leqslant)$ is a set A with a reflexive: $\boldsymbol{x \leqslant x}$ and transitive $\boldsymbol{x \leqslant y}$ and $\boldsymbol{y \leqslant z \implies x \leqslant z}$ binary relation.

A partially ordered set or **poset** $(A, \leqslant)$ is a preordered set that is antisymmetric: $\boldsymbol{x \leqslant y}$ and $\boldsymbol{y \leqslant x \implies x = y}$

A **semilattice** $(A, \cdot)$ is a semigroup that is commutative $\boldsymbol{x \cdot y = y \cdot x}$ and idempotent $\boldsymbol{x \cdot x = x}$

On a semilattice, define $x \leqslant y \iff \boldsymbol{x \cdot y = x}$

Lemma

$(A, \cdot)$ is a (meet-)semilattice $\iff$ $(A, \leqslant)$ is a poset and $x \cdot y =$ the greatest lower bound of x, y:
$\boldsymbol{x \cdot y \leqslant x, y}$ *and* $(\boldsymbol{z \leqslant x, y \Rightarrow z \leqslant x \cdot y})$

Exercise. Prove this lemma.

Lattices

A **lattice** $(A, \wedge, \vee)$ is a set with two associative, commutative and **absorptive** binary operations: $x \wedge (x \vee y) = x = x \vee (x \wedge y$

Lemma

$(A, \wedge, \vee)$ is a lattice $\iff$ $(A, \wedge)$, $(A, \vee)$ are semilattices and $x \wedge y = x \Leftrightarrow x \vee y = y$

Exercise 1. There are 16 four-element posets (up to isomorphism).

Here is a list of names to help with listing their **Hasse diagrams**:
1+1+1+1, 1+1+2, 1+V, 1+Λ, 2+2, 1+3, down, up, N, bowtie, Y, λ, J, Γ, diamond, 4

(a) Which ones are lattices?

(b) Which ones are meet-semilattices?

(c) Use this list to find all meet-semilattices with 5 elements.

(d) Use this list to find all lattices with 6 elements.
(The next results may help.)

A poset is **bounded** if it has a top ($\top$) and bottom ($\bot$) element.

Lemma

A finite meet-semilattice with $\top$ is a lattice.

A **doubly chopped lattice** is a poset $(A, \leqslant)$ such that $A \uplus \{\bot, \top\}$ is a lattice, where $\bot \leqslant x \leqslant \top$.

Lemma

Every lattice is a doubly chopped lattice.

Lemma

A poset is a doubly chopped lattice if and only if
$x, y \leqslant z, w \implies \exists u, (x, y \leqslant u \leqslant z, w)$.

A **residuated lattice** $(A, \wedge, \vee, \cdot, 1, /, \backslash)$ is a lattice $(A, \wedge, \vee)$ and a monoid such that $/$ is a right residual and $\backslash$ is a left residual:
$\boldsymbol{x \leqslant z/y \iff x \cdot y \leqslant z \iff y \leqslant x\backslash z}$

Some examples. $(\mathbb{R}, \min, \max, +, 0, -, -^{\mathrm{op}})$,

$([0,\infty], \min, \max, \cdot, 1, /, \backslash)$, where $x/y = y\backslash x = x{\div}y \ \ (x{\div}0 = \infty)$,

$([0,1], \min, \max, \cdot, 1, /, \backslash)$ where $x/y = y\backslash x = \min\{x \div y, 1\}$,

$([0,1], \min, \max, \cdot, 1, /, \backslash)$ where $x/y = y\backslash x = \min\{x \div y, 1\}$,

Algebras of binary relations $Rel(U) = (\mathcal{P}(U^2), \cap, \cup, \circ, id_U, /, \backslash)$,

where $R/S = \{(x,y) \in U^2 \mid \{(x,y)\} \circ S \subseteq R\}$.

Ward and Dilworth's original example (1939)

The ideal lattice of a commutative unital ring $(Id(R), \cap, \vee, \cdot, R, \backslash, /)$ where $I \cdot J = \langle\{i{\cdot}j \mid i \in I, j \in J\}\rangle$,

$I/J = J\backslash I = \{x \in R \mid x{\cdot}J \subseteq I\}$.

Exercise 2. Draw the lattice when $R = \mathbb{Z}_{12}$ and $\mathbb{Z}_{p^n}$.

Lemma

In a residuated lattice, $/$ is a right residual ($xy \leqslant z \Leftrightarrow x \leqslant z/y$) $\iff$ (1) $x \leqslant (xy \vee z)/y$ *and* (2) $(x \wedge (z/y))y \leqslant y$.

Proof.

Assume $xy \leqslant z \implies x \leqslant z/y$.

Then $xy \leqslant xy \vee z \implies x \leqslant (xy \vee z)/y$, so (1) holds.

Conversely assume (1). Then $xy \leqslant z \implies xy \vee z = z$

$\implies x \leqslant (xy \vee z)/y = z/y$, so $xy \leqslant z \implies x \leqslant z/y$ holds.

Exercise 3. Prove that $x \leqslant z/y \implies xy \leqslant z$ is equivalent to (2).

State, prove inequalities (3), (4) equivalent to $\backslash$ a left residual. □

Birkhoff's HSP theorem and Tarski's version

Let $\mathcal{K}$ be a class of algebras with the same basic operations.

$\mathbb{H}\mathcal{K} = \{$homomorphic images of members of $\mathcal{K}\}$

$\mathbb{S}\mathcal{K} = \{$subalgebras of members of $\mathcal{K}\}$

$\mathbb{P}\mathcal{K} = \{$direct products of members of $\mathcal{K}\}$

Theorem (Birkhoff 1935)

$\mathcal{K}$ is defined by a set of identities (= universally quantified equations) $\iff$ $\mathcal{K}$ is a **variety**, *i.e.*, $\mathbb{H}\mathcal{K} = \mathbb{S}\mathcal{K} = \mathbb{P}\mathcal{K} = \mathcal{K}$.

The smallest variety containing $\mathcal{K}$ is denoted $\mathbb{V}\mathcal{K}$.

Theorem (Tarski 1946)

The variety generated by $\mathcal{K}$ is $\mathbb{V}\mathcal{K} = \mathbb{H}\mathbb{S}\mathbb{P}\mathcal{K}$.

Residuated lattices are a variety

Corollary

The class RL *of all residuated lattices is defined by identities, hence it is a* **variety**, *i.e.,* $\mathbb{HSP}(\mathbf{RL}) = \mathbf{RL}$.

The smallest variety of residuated lattices is the class O of one-element RLs.

It is defined by the identity $x = y$.

The next smallest is $\mathbb{V}(\mathbf{2})$ where $\mathbf{2} = (\{0, 1\}, \cdot, \max, \cdot, 1, \to, \leftarrow)$ is the two element generalized Boolean algebra.

More properties of residuated lattices

Lemma

In a residuated lattice $\cdot$ distributes over $\vee$:

$x(y \vee z) = \mathbf{xy \vee xz}$ *and* $(x \vee y)z = \mathbf{xz \vee yz}$.

For right residuals:

$x/(y \vee z) = \mathbf{x/y \wedge x/z}$ *and* $(x \wedge y)/z = \mathbf{x/z \wedge y/z}$.

State and prove a similar result for left residuals.

Prove these results also hold for $\bot$ = empty join and $\top$ = empty meet (if they exist), i.e., $\bot \cdot x = \bot = x \cdot \bot$, $x/\bot = \top = \bot\backslash x$ and $\top/x = \top = x\backslash\top$.

In particular, $\bot/\bot = \top$, so a residuated lattice with bottom has a top element. Show that the converse can **fail** (Hint: $\mathbb{Z}^-$).

Prove that one can always add a new top or new bottom element to a residuated lattice.

Subvarieties of residuated lattices

A residuated lattices is **commutative** if $x \cdot y = y \cdot x$, **integral** if $x \leqslant 1$ and **idempotent** if $x \cdot x = x$

Exercise 4. Prove that in a commutative residuated lattice $x \backslash y = y/x$. We usually write $x \to y$ instead of $x \backslash y$.

Brouwerian algebras are idempotent integral residuated lattices. Prove that they are commutative.

Lemma

In a Brouwerian algebra, $x \cdot y = x \wedge y$, hence $\wedge$ distributes over $\vee$.

A (residuated) lattice is **distributive** if $\boldsymbol{x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)}$

Heyting algebras and intuitionistic logic

A residuated lattice is **pointed** if it has a constant symbol 0 (can refer to any element).

A **Heyting algebra** is a pointed Brouwerian algebra with 0 as bottom element: $\mathbf{0} \leqslant \boldsymbol{x}$

Inuitionistic negation is defined as $\neg x = \boldsymbol{x} \rightarrow \mathbf{0}$. Prove that $\neg 0 = 1$ and $\neg 1 = 0$.

Heyting algebras are the equivalent algebraic semantics of propositional **intuitionistic logic**

Boolean algebras from residuated lattices

A Boolean algebra is a Heyting algebra that satisfies **double-negation**: $\neg\neg x = x$

Boolean algebras are the equivalent algebraic semantics of propositional **classical logic**.

Substructural logic and residuated lattices

Residuated lattices give algebraic semantics for **substructural logics**

Structural rules are a concept from Gentzen's **proof theory**.

For residuated lattices, a structural rule is (roughly) a universally quantified implication that uses only the operation $\cdot$ and 1, e.g.,

contraction: $u{\cdot}x{\cdot}v \leqslant w \implies u{\cdot}x{\cdot}x{\cdot}v \leqslant w$, $\quad \dfrac{\Gamma,X,\Pi \vdash \Delta}{\Gamma,X,X,\Pi \vdash \Delta}(\text{c})$

exchange: $u{\cdot}x{\cdot}y{\cdot}v \leqslant w \implies u{\cdot}y{\cdot}x{\cdot}v \leqslant w$, $\quad \dfrac{\Gamma,X,Y,\Pi \vdash \Delta}{\Gamma,Y,X,\Pi \vdash \Delta}(\text{e})$

weakening: $u{\cdot}x{\cdot}v \leqslant w \implies u{\cdot}x{\cdot}y{\cdot}v \leqslant w$, $\quad \dfrac{\Gamma,X,\Pi \vdash \Delta}{\Gamma,X,Y,\Pi \vdash \Delta}(\text{w})$

cut: $v \leqslant x$ and $u{\cdot}x{\cdot}w \leqslant z \implies u{\cdot}v{\cdot}w \leqslant z$, $\quad \dfrac{\Lambda \vdash X \quad \Gamma,X,\Pi \vdash \Delta}{\Gamma,\Lambda,\Pi \vdash \Delta}(\text{cut})$

For RL these rules are $x{\cdot}x \leqslant x$, $x{\cdot}y = y{\cdot}x$, $x \leqslant 1$, transitivity.

Gödel algebras and (G)BL-algebras

A class K is **semilinear** if every member is a subalgebra of a product of linearly ordered members in K.

A commutative residuated lattice is **prelinear** if $1 \leqslant x/y \vee y/x$ holds. **Exercise 6.** Prove it is semilinear.

A **Gödel algebra** is a prelinear Heyting algebra.

A residuated lattice is **divisible** if $x \leqslant y \Rightarrow x = (x/y)y = y(y\backslash x)$.

A **generalized basic logic algebra** (or GBL-algebra) is a divisible residuated lattice.

Exercise 7. Prove that GBL-algebras are a variety, and are distributive.

A **basic logic algebra** (or BL-algebra) is a bounded commutative integral prelinear GBL-algebra.

MV-algebras and ℓ-groups

An **involutive residuated lattice** (or InRL) is a pointed residuated lattice where $(0/x)\backslash 0 = 0/(x\backslash 0)$ holds.

A **multi-valued algebra** (or MV-algebra) is an involutive BL-algebra.

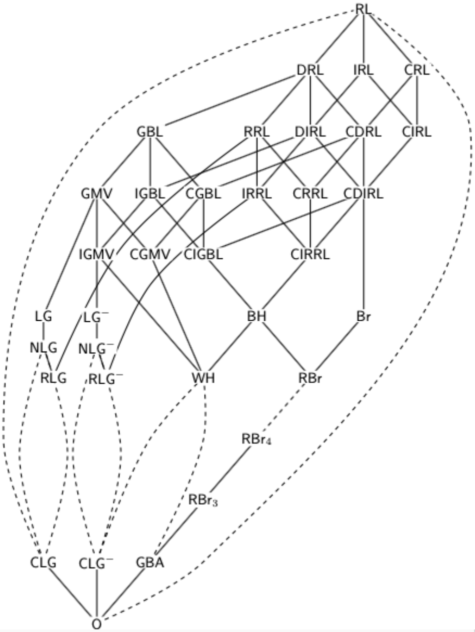
The standard MV-algebra: $([0,1], \min, \max, \cdot, 1, \to, 0)$ where $x{\cdot}y = \max\{x+y-1, 0\}$, finite subalgebras $\text{Ł}_n = \{\frac{0}{n-1}, \frac{1}{n-1}, \ldots, 1\}$

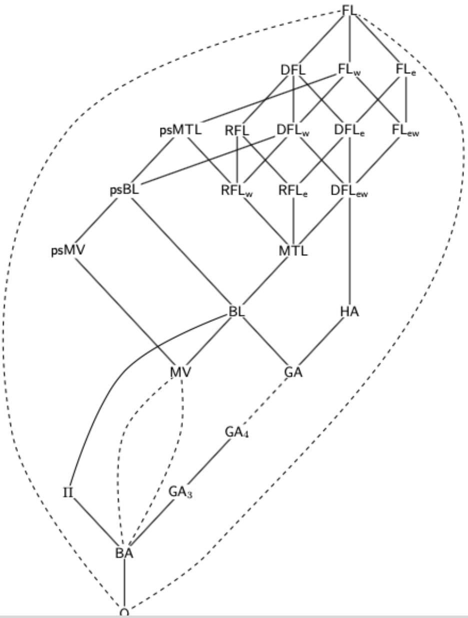
A **lattice-ordered group** (or ℓ-group) is a residuated lattice that satisfies $x(1/x) = 1$. Then $1/x = x\backslash 1$ and is denoted x^{-1}.

An **abelian** ℓ-groups is commutative. E.g., $(\mathbb{Z}, \max, \min, +, 0, -)$
Exercise 8. Prove that ℓ-groups are distributive.

Note: The varieties above correspond to Gödel logic, Hajek's basic logic, Łukasiewicz logic and abelian logic.

Subvarieties of residuated lattices and pointed RL





Congruences of residuated lattices

Residuated lattices are 1-**regular**, and have distributive congruence lattices (since they have lattice reducts).

A **normal filter** F is a subset of a residuated lattice where $\{1\},\ \uparrow F,\ F{\cdot}F,\ (x{\cdot}F)/x,\ x\backslash(F{\cdot}x) \subseteq F$.

Theorem

There is an order-preserving bijection between congruences and normal filters via the maps

$\theta \mapsto F_\theta = \uparrow[1]_\theta$ *and* $F \mapsto \theta_F = \{(x,y) \mid x/y, y/x \in F\}$.

For finite RL, $\mathrm{Con}(A) \cong$ *negative central idempotents.*

Residuated frames

Residuated lattices have Kripke semantics given by residuated frames:

$(W, W', N, \circ, E, /\!\!/, \backslash\!\!\backslash)$ where N a **nuclear relation** $N \subseteq W \times W'$,

$\circ \subseteq W^3$, $E \subseteq W$, $/\!\!/ \subseteq W' \times W \times W'$, $\backslash\!\!\backslash \subseteq W \times W' \times W'$ such that

$$X\ N\ (Z /\!\!/ Y) \iff (X \circ Y)\ N\ Z \iff Y\ N\ (X \backslash\!\!\backslash Z).$$

Let $\gamma_N(X) = \{z \mid \forall y((\forall x \in X, (xNy)) \Rightarrow zNy\}$. Then $\gamma_N[\mathcal{P}(W)]$ is a complete residuated lattice-ordered groupoid.

Provides an algebraic way to prove cut-elimination.

The equational theory of RL is decidable by a cut-free Genten system that has the subformula property.

Further reading

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- N. Galatos, P. Jipsen, T. Kowalski, H. Ono: Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Studies in Logic and the Foundations of Mathematics, Vol. 151, Elsevier, (2007) xxiv+509.
- P. Jipsen, F. Montagna: The Blok-Ferreirim theorem for normal GBL-algebras and its application, Alg. Univ., 60, (2009), 381–404.
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- S. Bonzio, J. Gil-Férez, P. Jipsen, A. Prenosil, M. Sugimoto: On the structure of balanced residuated partially-ordered monoids, preprint

Thanks!