Plonka sums of metamorphisms applied to balanced residuated posets

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Logics of Variable Inclusion

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Residuated posets

A **residuated poset** is a structure $\mathbf{A} = (A, \leqslant, \cdot, 1, /, \backslash)$ such that

- $(A, \leqslant)$ is a poset,
- $(A, \cdot, 1)$ is a monoid,
- $xy \leqslant z \iff x \leqslant z/y \iff y \leqslant x\backslash z$ **(res)**.

$\implies$ $\cdot$ is order-preserving in both arguments

$/, \backslash$ are order-preserving in numerator, order-reversing in denominator

Some examples: $(\mathbb{R}, \leqslant, +, 0, -, -^{\mathrm{op}})$, any ℓ-**group**,

all **groups** (where $\leqslant$ is $=$, $x\backslash y = x^{-1}y$ and $x/y = xy^{-1}$)

all $\wedge, \vee$-free reducts of **residuated lattices**.

The **algebraic semantics of any logic** with a fusion, a truth constant and two implications with a deduction theorem (where $\leqslant$ is $\vdash$)

Balanced residuated posets

A residuated poset is **balanced** if $x/x = x\backslash x$

idempotent if $x^2 = x$

integral if $x \leqslant 1$

Example: every commutative residuated lattice is balanced,

every Boolean algebra is idempotent and integral
(since $xy = x \wedge y$, so $xx = x$, and $x1 = x$ implies $x \leqslant 1$).

A **residuated lattice** is a residuated poset that is a lattice (has $\vee, \wedge$).

The aim of this talk

Decompose (certain) residuated posets into **simpler components**.

The components are residuated posets with a **unique positive idempotent**.

Reconstruction uses **Płonka sums of metamorphisms**.

Extends structure theory for **even/odd involutive $\mathbf{FL}_e$-chains** [Jenei 2022],

finite commutative idempotent involutive residuated lattices, the components are **Boolean algebras** [Jipsen, Tuyt, Valota 2021],

and **locally integral involutive residuated posets**, where the components are **integral** involutive residuated posets [Gil-Férez, Jipsen, Lodhia 2023].

How many small residuated posets are there?

Cardinality $n =$	1	2	3	4	5	6	7	8	9
Residuated posets (RP)	1	2	5	28	186	1795			
Residuated lattices (RL)	1	1	3	20	149	1488	18554	295292	
Balanced RP	1	2	5	26	184	1461			
Unique pos. idem. RP	1	2	4	16	72	516			
Integral RP	1	1	2	9	49	365	3348		
Idempotent RP	1	1	2	8	32	148	785		
Comm. idempotent RP	1	1	2	6	20	78	351	1798	
Idempotent integral RP	1	1	1	2	3	5	8	15	26

A residuated poset is **commutative** if $xy = yx$.

Can we find good structure theory for some of these classes?

Examples of good structure theory

Cardinality $n =$	1	2	3	4	5	6	7	8	9	10	11	12
Boolean algebras	1	1	0	1	0	0	0	1	0	0	0	0
Fields	1	1	1	1	1	0	1	1	1	0	1	0
Abelian groups	1	1	1	2	1	1	1	3	2	1	1	2
MV-algebras	1	1	1	2	1	2	1	3	2	2	1	4
Gödel algebras	1	1	1	2	2	3	3	5	6	8	8	12
Distr. lattices	1	1	1	2	3	5	8	15	26	47	82	151
BL-algebras	1	1	2	5	9	20	38	81	160	326	643	1314

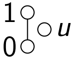
Note that all these examples are commutative.

Cardinality $n =$	1	2	3	4	5	6	7	8
Partially ordered monoids	1	6	109	5849				
Residuated posets (RP)	1	2	5	28	186	1795		
Residuated lattices	1	1	3	20	149	1488	18554	295292
Idempotent integral RP	1	1	1	2	3	5	8	15

Small residuated posets

There is a unique residuated poset (RP) of cardinality 1.

There are 2 RP of cardinality 2: the BA $\mathbf{2}$ and the 2-elt group $\mathbb{Z}_2$

There are 5 RP of cardinality 3: $\mathbf{G}_3$, $\mathbf{Ł}_3$, $\mathbf{S}_3$, $\mathbb{Z}_3$ and $\mathbf{2}{+}\mathbf{1}$ 

$0x = x0 = 0$, $1x = x1 = x$, $u^2 = 0$

There are 28 RP of cardinality 4: 15 are linearly ordered,
5 are based on $\mathbf{2}{\times}\mathbf{2}$, 3 are based on $\mathbf{3}{+}\mathbf{1}$, 2 are based on $\mathbf{2}{+}\mathbf{2}$,
$\mathbf{2}{+}\mathbf{1}{+}\mathbf{1}$, $\mathbb{Z}_4$, $\mathbb{Z}_2{\times}\mathbb{Z}_2$

A list of small residuated lattices

RESIDUATED LATTICES OF SIZE UP TO 6

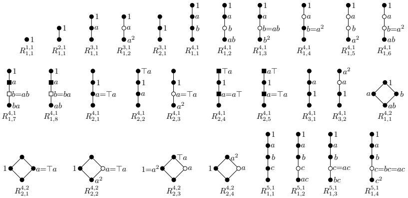
NICK GALATOS AND PETER JIPSEN

There are $1+1+3+20+149+1488 = 1662$ residuated lattices with ≤ 6 elements. In the list below, each algebra is named R^{mn}_{ij} where m is the cardinality and n enumerates nonisomorphic lattices of size m, in order of decreasing height. The depth of the identity element 1 is given by i, and j enumerates nonisomorphic algebras. For lattices of the same height distributive lattices appear before modular lattices, followed by nonmodular lattices, and selfdual lattices appear before nonselfdual lattices. Algebras with more central elements (round circles) are listed earlier, hence commutative residuated lattices precede noncommutative ones. Finally, algebras are listed in decreasing order of number of idempotents (black nodes).

The monoid operation is indicated by labels. If a nonobvious product xy is not listed, then it can be deduced from the given information: either it follows from idempotence ($x^2 = x$) indicated by a black node or from commutativity or there are products $uv = wz$ such that $u \leq x \leq w$ and $v \leq y \leq z$ (possibly $uv = \bot\bot$ or $wz = \top\top$).

If you have comments or notice any issues in this list, please email `jipsen.AT.chapman.edu`.

● = central idempotent
○ = central nonidempotent
■ = noncentral idempotent
□ = noncentral nonidempotent



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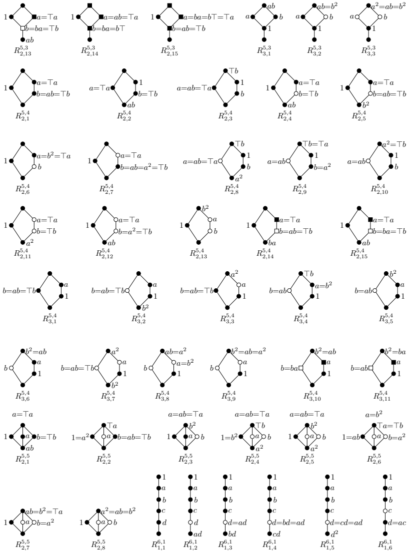


Positive idempotent elements are central

Lemma

In a balanced residuated poset any idempotent positive element p is central, i.e., for every x, $px = xp$.

Proof.

Assume $x/x = x\backslash x$ holds in $\mathbf{A}$ and let $p \in Id^+(\mathbf{A})$

Then $ppx \leqslant px$ since $pp = p$.

$\implies p \leqslant px/px = px\backslash px$ by (res) and balanced

$\implies pxp \leqslant px$ by (res)

$x = 1x \leqslant px$ since p is positive and $\cdot$ order-preserving

$\implies xp \leqslant pxp$ again by order-preservation.

Therefore $xp \leqslant px$.

Similarly $px \leqslant xp$, so p is central. $\square$

Positive idempotents

Let $Id^+(\mathbf{A}) = \{p \in A \mid 1 \leqslant p = p^2\}$ = the set of positive idempotent elements of $\mathbf{A}$.

Lemma

If $\mathbf{A}$ is a balanced residuated poset then $(Id^+(\mathbf{A}), \cdot, 1)$ is a join-semilattice with bottom 1 and the order of $\mathbf{A}$ agrees with the join-semilattice order on $Id^+(\mathbf{A})$.

Proof.

Positive idempotents are closed under $\cdot$, and this operation is associative, commutative (by the previous lemma) and idempotent.

If $p, q \in Id^+(\mathbf{A})$ and $p \leqslant q$ in $\mathbf{A}$ then $q = 1q \leqslant pq$ since $1 \leqslant p$.

$pq \leqslant qq = q$ since $\cdot$ is order-preserving and q is idempotent.

Hence $pq = q$, which means $p \leqslant q$ in $Id^+(A)$. □

Positive idempotents

Lemma

In a residuated poset $\mathbf{A}$, $Id^+(\mathbf{A}) = \{x/x \mid x \in A\} = \{x\backslash x \mid x \in A\}$.

Proof.

In any residuated poset $(y/z)z \leqslant y$, so

$(x/x)(x/x)x \leqslant (x/x)x \leqslant x \implies (x/x)(x/x) \leqslant x/x$.

$1 \leqslant x/x$, so x/x is idempotent, hence $\{x/x : x \in A\} \subseteq Id^+(\mathbf{A})$.

$x \in Id^+(\mathbf{A}) \implies x \cdot x = x \implies x \leqslant x/x$ and

$1 \leqslant x \implies x/x \leqslant x/1 = x$, so $x/x = x$.

Therefore $Id^+(\mathbf{A}) \subseteq \{x/x : x \in A\}$. □

Corollary

A residuated poset satisfies $\forall x, x/x = 1 \iff \forall x, x\backslash x = 1$.

Left local identities

$$x/x \leqslant x/x \implies (x/x)x \leqslant x$$

$$1x \leqslant x \implies 1 \leqslant x/x \implies x = 1x \leqslant (x/x)x$$

Therefore $(x/x)x = x$ in any residuated poset.

x/x is called a **left local identity** and is denoted 1_x.

Lemma

In a residuated poset $\mathbf{A}$, 1_x *is an identity for* x *iff* $x/x = x\backslash x$.

An important equivalence relation

Define $x \equiv y \iff 1_x = 1_y$.

Then $\equiv$ is an equivalence relation.

Let $A_x = \{y \in A \mid 1_x = 1_y\}$ be the equivalence classes.

For a residuated poset $\mathbf{A}$ the equivalence classes A_x are called the **local components** of $\mathbf{A}$.

Decomposable residuated posets

A residuated poset is **decomposable** if $1_x = 1_y \implies 1_{x\backslash y} = 1_x$.

Lemma

A decomposable residuated poset is balanced and satisfies

$$1_x = 1_y \implies 1_{xy} = 1_x \quad \textit{and} \quad 1_x = 1_y \implies 1_{x/y} = 1_x.$$

Examples: All **commutative idempotent** residuated posets.

A RP is **involutive** if $0/(x\backslash 0) = (0/x)\backslash 0$ for some constant 0.

[Gil-Férez, J., Lodhia] all locally integral **involutive** residuated posets are decomposable.

The decomposition theorem

Theorem

Every decomposable residuated poset $\mathbf{A}$ *is a disjoint union of residuated posets* $\mathbf{A}_p = (A_p, \leqslant_p, \cdot_p, 1_p, \backslash_p, /_p)$ *where*
p ranges over the join-semilattice $(Id^+(\mathbf{A}), \leqslant, \cdot)$,
$p = 1_p$ *is the unique positive idempotent of* $\mathbf{A}_p$ *and*
$\leqslant_p, \cdot_p, \backslash_p, /_p$ *are the restrictions of* $\leqslant, \cdot, \backslash, /$ *to* A_p*.*

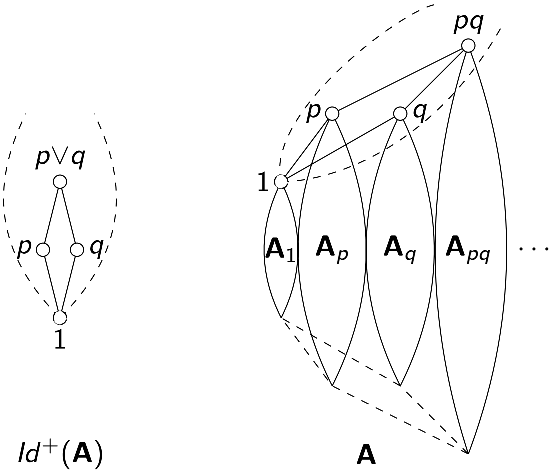
Proof.

Suppose $\mathbf{A}$ is decomposable and let $p \in Id^+(\mathbf{A})$, so $p = 1_p$.
Then $\mathbf{A}$ is balanced, so p is a left and right identity on A_p.
For $x, y \in A_p$ we have $1_x = p = 1_y$,
so the preceding lemma implies $1_{xy} = 1_x$ and $1_{x/y} = 1_x = p$.
Therefore $xy, x/y \in A_p$, so the $\mathbf{A}_p$ are residuated posets. □

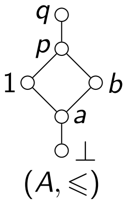
A diagram of the decomposition into local components

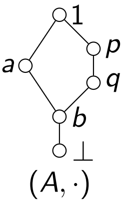
Each component A_p intersects $Id^+(\mathbf{A})$ in a **unique** element $1_x = p$.

The $\mathbf{A}_p$ are **integral** $\iff$ $\mathbf{A}$ is **square-decreasing** ($x \cdot x \leqslant x$).



A six-element decomposable example





The left poset $(A, \leqslant)$ can be equipped with a commutative idempotent multiplication, i.e., the meet operation of the right poset.

This multiplication preserves all joins of $(A, \leqslant)$, hence is residuated.

This gives a residuated poset $\mathbf{A}$, where $Id^+(\mathbf{A}) = \{1, p, q\}$ and $A_1 = \{1, a\}$, $A_p = \{p\}$, and $A_q = \{q, b, \perp\}$.

These sets are closed under residuals, hence $\mathbf{A}$ is decomposable.

Reconstructing RPs from local components

In many cases the original residuated poset can be reconstructed from the local components and two families of maps:

$\varphi_{pq}, \psi_{pq} : \mathbf{A}_p \to \mathbf{A}_q$ for $p \leqslant q \in Id^+(\mathbf{A})$.

The maps are defined by $\varphi_{pq}(x) = qx$ and $\psi_{pq}(x) = q\backslash x$.

Reconstructing the monoid operation uses a **Płonka sum**.

Reconstructing the order and residuals requires a generalization.

Brief review of Płonka sums

Let $\mathbf{I} = (I, \vee)$ be a join semilattice of **indices**.

A **semilattice directed system** $\Phi = \{\varphi_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j : i \leqslant j \text{ in } \mathbf{I}\}$ is a family of homomorphisms between algebras of the same type if φ_{ii} is the identity on $\mathbf{A}_i$ and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, for all $i \leqslant j \leqslant k$.

If the algebras contain constants, assume $\mathbf{I}$ has a least element $\perp$.

The **Płonka sum** of a semilattice directed system Φ is an algebra $\mathbf{S}$ of the same type defined on the disjoint union of the universes $S = \biguplus_{i \in I} A_i$.

For every n-ary operation symbol σ and $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$,

$$\sigma^{\mathbf{S}}(a_1, \ldots, a_n) = \sigma^{\mathbf{A}_j}(\varphi_{i_1 j}(a_1), \ldots, \varphi_{i_n j}(a_n)),$$

where $j = i_1 \vee \cdots \vee i_n$, and for every constant symbol ω, $\omega^{\mathbf{S}} = \omega^{\mathbf{A}_\perp}$.

Partition functions

A **partition function** on an algebra $\mathbf{A}$ is a binary operation $\odot\colon A^2 \to A$ satisfying the following conditions, for every n-ary operation symbol σ, every constant symbol ω, and all $a, b, c, a_1, \ldots, a_n \in A$.

(PF1) $a \odot a = a$,

(PF2) $a \odot (b \odot c) = (a \odot b) \odot c$,

(PF3) $a \odot (b \odot c) = a \odot (c \odot b)$,

(PF4) $\sigma^{\mathbf{A}}(a_1, \ldots, a_n) \odot b = \sigma^{\mathbf{A}}(a_1 \odot b, \ldots, a_n \odot b)$,

(PF5) $b \odot \sigma^{\mathbf{A}}(a_1, \ldots, a_n) = b \odot a_1 \odot \cdots \odot a_n$,

(PF6) $b \odot \omega^{\mathbf{A}} = b$.

Płonka's decomposition theorem

Let $\mathbf{A}$ be an algebra with a partition function $\odot$.

1. There exists a partition $\{A_i : i \in I\}$ of A, such that any two elements $a, b \in A$ belong to the same equivalence class exactly when $a \odot b = a \quad \text{and} \quad b \odot a = b$.
2. The relation $\leqslant$ on I defined by the condition $i \leqslant j \iff \exists a \in A_i, b \in A_j$ such that $b \odot a = b$ is a partial order and $\mathbf{I} = (I, \leqslant)$ is a join semilattice. If $\mathbf{A}$ contains some constant, then $\mathbf{I}$ has a least element $\bot$.
3. Each A_i is the universe of an algebra $\mathbf{A}_i$ of the same type, whose constant-free reduct is a subalgebra of the constant-free reduct of $\mathbf{A}$. If $\bot \in I$ then $\mathbf{A}_\bot$ is a subalgebra of $\mathbf{A}$.
4. For all $i, j \in I$ such that $i \leqslant j$ there is a homomorphism $\varphi_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j$, defined by $\varphi_{ij}(a) = a \odot b$, for any $b \in A_j$.
5. $\mathbf{A}$ is the Płonka sum of the semilattice directed system $\{\varphi_{ij} : i \leqslant j \text{ in } \mathbf{I}\}$.

Partition system for an algebra

A **left normal band** is a binary operation that satisfies (PF1)-(PF3).

A set of left normal bands is **compatible** if they induce the same equivalence relation.

A **partition system** for an algebra $\mathbf{A}$ is an assignment $\sigma \mapsto (\odot_0^\sigma, \ldots, \odot_n^\sigma)$ for every n-ary operation or constant symbol σ of the type of $\mathbf{A}$ to tuples of elements of a set O of **compatible left normal bands** on A, such that for every n-ary operation symbol σ, every constant symbol ω, and all $a_1, \ldots, a_n, b \in A$,

(PF4$^\sigma$) $\sigma^{\mathbf{A}}(a_1, \ldots, a_n) \odot_0^\sigma b = \sigma^{\mathbf{A}}(a_1 \odot_1^\sigma b, \ldots, a_n \odot_n^\sigma b)$,

(PF5$^\sigma$) $b \odot_0^\sigma \sigma^{\mathbf{A}}(a_1, \ldots, a_n) = b \odot_0^\sigma a_1 \odot_0^\sigma \cdots \odot_0^\sigma a_n$,

(PF6$^\omega$) $b \odot_0^\omega \omega^{\mathbf{A}} = b$.

Metamorphisms between algebras

Let $\mathbf{A}, \mathbf{B}$ be algebras with operation symbols $\sigma \in \mathcal{O}$.

A **metamorphism** $h : \mathbf{A} \looparrowright \mathbf{B}$ is a sequence of functions $h^\sigma = (h^{\sigma 0}, \ldots, h^{\sigma n})$ for each n-ary symbol $\sigma \in \mathcal{O}$ such that

$$h^{\sigma 0}(\sigma^{\mathbf{A}}(a_1, \ldots, a_n)) = \sigma^{\mathbf{B}}(h^{\sigma 1}(a_1), \ldots, h^{\sigma n}(a_n)).$$

In particular, for every constant ω, $f^\omega = (f^{\omega 0})$ and $f^{\omega 0}(\omega^{\mathbf{A}}) = \omega^{\mathbf{B}}$.

Every homomorphism $g : \mathbf{A} \to \mathbf{B}$ gives rise to a **metamorphism** $h^\sigma = (g, \ldots, g)$, and algebras of the same type form a category with componentwise composition of metamorphisms:

If $h\colon \mathbf{A} \looparrowright \mathbf{B}$ and $k\colon \mathbf{B} \looparrowright \mathbf{C}$ then $k \circ h\colon \mathbf{A} \looparrowright \mathbf{C}$ is defined by $(k \circ h)^\sigma = (k^{\sigma 0} \circ h^{\sigma 0}, \ldots, k^{\sigma n} \circ h^{\sigma n})$.

The **identity** metamorphism is $id\colon \mathbf{A} \looparrowright \mathbf{A}$, $id^\sigma = (id^A, \ldots, id^A)$.

Płonka sum of metamorphisms

A **semilattice directed system of metamorphisms** is a family $\mathcal{H} = \{h_{pq} \colon \mathbf{A}_p \nrightarrow \mathbf{A}_q : p \leqslant q \text{ in } \mathbf{I}\}$ such that $h_{pp} = id^{\mathbf{A}_p}$ and $h_{qr} \circ h_{pq} = h_{pr}$.

The **Płonka sum of a directed system of metamorphisms** $\mathcal{H}$ is the algebra $\mathbf{A}$ whose universe is $A = \biguplus A_p$, and such that for every n-ary operation σ and all $a_1 \in A_{p_1}, \ldots, a_n \in A_{p_n}$,

$$\sigma^{\mathbf{A}}(a_1, \ldots, a_n) = \sigma^{\mathbf{A}_q}(h^{\sigma 1}_{p_1 q}(a_1), \ldots, h^{\sigma n}_{p_n q}(a_n)),$$

where $q = p_1 \vee \cdots \vee p_n$.

If the type τ contains a constant symbol ω, then we assume that $\mathbf{I}$ has a least element $\perp$ and $\omega^{\mathbf{A}} = \omega^{\mathbf{A}_\perp}$.

Theorem

Let $\mathbf{A}$ *be an algebra with a partition system,* $\equiv$ *the induced equivalence relation and* $\mathbf{I} = (I, \vee)$ *the induced join semilattice.*

1. *Every equivalence class* A_p *of* $\equiv$ *is the universe of an algebra* $\mathbf{A}_p$ *of the same type whose constant-free reduct is a subalgebra of the constant-free reduct of* $\mathbf{A}$*, and* $\omega^{\mathbf{A}_p} = \omega^{\mathbf{A}} \odot_0^\omega p$ *for every constant symbol* ω*.*
2. *For all* $p \leqslant q$ *in* $\mathbf{I}$*, there is a metamorphism* $h_{pq} \colon \mathbf{A}_p \looparrowright \mathbf{A}_q$ *defined by*
$$h_{pq}^{\sigma i}(a) = a \odot_i^\sigma q, \quad \text{for every } n\text{-ary symbol } \sigma \text{ and } i \leqslant n.$$
3. *The family* $\mathcal{H} = \{h_{pq} \colon \mathbf{A}_p \looparrowright \mathbf{A}_q : p \leqslant q \text{ in } \mathbf{I}\}$ *is a semilattice directed system of metamorphisms and* $\mathbf{A}$ *is its Płonka sum.*

Płonka decomposition using metamorphisms

Theorem

Every semilattice directed system of metamorphisms is induced by a partition system for its Płonka sum $\mathbf{A}$*.*

A residuated poset is **Płonka summable** if it satisfies
$1_{xy} = 1_{x\backslash y} = 1_{x/y} = 1_x \cdot 1_y$.

Theorem

Let $\mathbf{A}$ *be a Płonka summable residuated poset and* $O = \{\odot, \otimes\}$ *left normal bands defined by* $a \odot b = 1_b \cdot a$ *and* $a \otimes b = 1_b \backslash a$*.*

Then assigning $1 \mapsto (\odot)$, $\cdot \mapsto (\odot, \odot, \odot)$, $\backslash \mapsto (\otimes, \odot, \otimes)$, *and* $/ \mapsto (\otimes, \otimes, \odot)$ *defines a partition system for the algebraic reduct of* $\mathbf{A}$*.*

Reconstructing Płonka summable residuated posets

Recall that for $p \leqslant q \in Id^{+}(\mathbf{A})$ we defined $\varphi_{pq}, \psi_{pq} : \mathbf{A}_p \to \mathbf{A}_q$ by

$$\varphi_{pq}(x) = qx \quad \text{and} \quad \psi_{pq}(x) = q\backslash x.$$

Corollary

The algebraic reduct of a Płonka summable residuated poset $\mathbf{A}$ is the Płonka sum of the directed system of metamorphisms $\mathcal{H} = \{h_{pq} \colon \mathbf{A}_p \nrightarrow \mathbf{A}_q : p \leqslant q \text{ in } \mathbf{I}\}$ given by

$$h^{1}_{pq} = (\varphi_{pq}), \qquad h^{\cdot}_{pq} = (\varphi_{pq}, \varphi_{pq}, \varphi_{pq}),$$
$$h^{\backslash}_{pq} = (\psi_{pq}, \varphi_{pq}, \psi_{pq}), \qquad h^{/}_{pq} = (\psi_{pq}, \psi_{pq}, \varphi_{pq}).$$

Moreover, for all $p, q \in I$, $a \in A_p$, and $b \in A_q$,

$$a \leqslant b \iff \varphi_{ps}(a) \leqslant_s \psi_{qs}(b), \quad \textit{where } s = pq.$$

Sums of posets

Let $\mathbf{I} = (I, \vee)$ be a join-semilattice and (Φ, Ψ) a pair of directed systems of monotone maps $\Phi = \{\varphi_{pq}\colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant q \text{ in } \mathbf{I}\}$ and $\Psi = \{\psi_{pq}\colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant q \text{ in } \mathbf{I}\}$.

Define the relation $\leqslant$ on $A = \biguplus A_p$ as follows: for all $p, q \in I$, $a \in A_p$, and $b \in A_q$,

$$a \leqslant b \quad \iff \quad \varphi_{ps}(a) \leqslant_s \psi_{qs}(b), \quad \text{where } s = p \vee q.$$

In general, this relation is not a partial order, but it will be if the following three conditions are satisfied. In that case, we call $(A, \leqslant)$ the **sum** of the family of posets $\{\mathbf{A}_p : p \in I\}$ **over** (Φ, Ψ).

(O1) if $p < q$ then $\psi_{pq} <_q \varphi_{pq}$ pointwise,

(O2) if $p \leqslant q, r$ and $t = q \vee r$, then $\varphi_{qt}\psi_{pq} \leqslant_t \psi_{rt}\varphi_{pr}$ pointwise,

(O3) for all $a, b \in A_p$ and $p \leqslant q$, if $\varphi_{pq}(a) \leqslant_q \psi_{pq}(b)$, then $a \leqslant_p b$.

Construction of posets from components

Theorem

Given a pair (Φ, Ψ) of directed systems of monotone maps, the relation $\leqslant$ defined by

$$a \leqslant b \quad \iff \quad \varphi_{ps}(a) \leqslant_s \psi_{qs}(b), \quad \textit{where } s = p \vee q.$$

is a partial order extending the order of each poset if and only if (Φ, Ψ) satisfies (O1)–(O3).

Construction Theorem

Let $\{\mathbf{A}_p\colon p \in I\}$ be a family of residuated posets indexed on a join semilattice $\mathbf{I} = (I, \vee)$ with least element $\perp$ and Φ, Ψ a pair of semilattice directed systems of monotone maps such that $\mathcal{H} = \{h_{pq}\colon \mathbf{A}_p \looparrowright \mathbf{A}_q : p \leqslant q \text{ in } \mathbf{I}\}$ is a directed system of metamorphisms defined by

$$h^1_{pq} = (\varphi_{pq}), \qquad h^{\cdot}_{pq} = (\varphi_{pq}, \varphi_{pq}, \varphi_{pq}),$$
$$h^{\backslash}_{pq} = (\psi_{pq}, \varphi_{pq}, \psi_{pq}), \qquad h^{/}_{pq} = (\psi_{pq}, \psi_{pq}, \varphi_{pq}).$$

and (Φ, Ψ) satisfies (O1)–(O3). Then the Płonka sum of $\mathcal{H}$ together with the sum of the poset reducts over (Φ, Ψ) is a residuated poset.

Application to involutive residuated posets

An **involutive residuated poset**, or InRP, is a structure of the form $\mathbf{A} = (A, \leqslant, \cdot, 1, \sim, -)$ such that $(A, \leqslant)$ is a poset and $(A, \cdot, 1)$ is a monoid satisfying

$$x \leqslant y \iff x \cdot {\sim}y \leqslant -1 \iff -y \cdot x \leqslant -1. \qquad \text{(ineg)}$$

The monoid operation is residuated with residuals defined by $x \backslash y = {\sim}(-y \cdot x)$ and $x/y = -(y \cdot {\sim}x)$.

An ipo-monoid $\mathbf{A}$ is **locally integral** if it is balanced, and it satisfies $x \leqslant 1_x$ and $x \backslash 1_x = 1_x$, where $1_x = x/x = -(x \cdot {\sim}x)$.

Define $\odot, \otimes \colon A^2 \to A$ by $a \odot b = 1_b \cdot a$ and $a \otimes b = {\sim}(-a \cdot 1_b)$.

Then $\odot, \otimes$ are compatible left normal bands and the assignment $1 \mapsto \odot$, $\cdot \mapsto (\odot, \odot, \odot)$, $\sim \mapsto (\otimes, \odot)$, and $- \mapsto (\otimes, \odot)$ is a partition system for $\mathbf{A}$.

Płonka sums of InRP

The metamorphisms are determined by the two families of maps

$$\varphi_{pq}(x) = x \odot q = 1_q \cdot x = q \cdot x \text{ and}$$

$$\psi_{pq}(x) = x \otimes q = {\sim}(-x \cdot 1_q) = q \backslash x.$$

Hence every locally integral InRP is a Płonka sum, and its order can be recovered by

$$a \leqslant b \quad \iff \quad \varphi_{ps}(a) \leqslant_s \psi_{qs}(b), \quad \text{where } s = p \vee q.$$

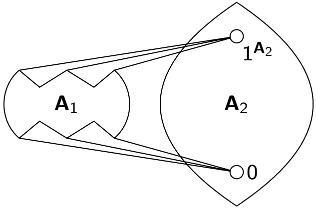
This is the structure theory originally obtained in an ad hoc manner in [Gil-Férez, J., Lodhia 2023].

Example: Płonka sum of two residuated posets

Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two residuated posets, with two directed systems $\Phi = \{\varphi_{pq} : p \leqslant q\}$ and $\Psi = \{\psi_{pq} : p \leqslant q\}$, indexed over the 2-element chain $1 < 2$, such that the nonidentity maps $\varphi_{12}, \psi_{12} \colon A_1 \to A_2$ are defined by

$$a \mapsto \varphi_{12}(a) = 1^{\mathbf{A}_2} \quad \text{and} \quad a \mapsto \psi_{12}(a) = 0,$$

with 0 a fixed element in A_2 such that $0 < 1^{\mathbf{A}_2}$.



Then (Φ, Ψ) satisfies the conditions of the **Construction Theorem**, hence we obtain a residuated poset $\mathbf{S} = \mathbf{A}_1 \uplus \mathbf{A}_2$.

When will the construction produce a residuated lattice?

In general this is an open problem, but for the two-component Płonka sum it suffices if $\mathbf{A}_1$ is a (chopped) lattice and $\mathbf{A}_2$ is a lattice.

Join and meet are defined as follows:

$$a \vee^{\mathbf{S}} b = \begin{cases} a \vee^{\mathbf{A}} b & \text{if } a, b \in A \text{ have an upper bound in } A \\ 1^{\mathbf{B}} & \text{if } a, b \in A \text{ have no upper bound in } A \\ a \vee^{\mathbf{B}} b & \text{if } a, b \in B \\ a & \text{if } a \in A,\ b \in B \text{ with } b \leqslant 0 \\ b \vee^{\mathbf{B}} 1^{\mathbf{B}} & \text{if } a \in A,\ b \in B \text{ with } b \nleqslant 0, \end{cases}$$

$$a \wedge^{\mathbf{S}} b = \begin{cases} a \wedge^{\mathbf{A}} b & \text{if } a, b \in A \text{ have a lower bound in } A \\ 0 & \text{if } a, b \in A \text{ have no lower bound in } A \\ a \wedge^{\mathbf{B}} b & \text{if } a, b \in B \\ a & \text{if } a \in A,\ b \in B \text{ with } 1^{\mathbf{B}} \leqslant b \\ b \wedge^{\mathbf{B}} 0 & \text{if } a \in A,\ b \in B \text{ with } 1^{\mathbf{B}} \nleqslant b. \end{cases}$$

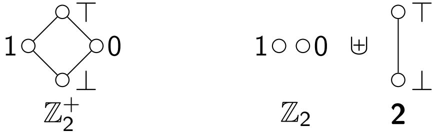
Example of a 4-element relation algebra

For any monoid $\mathbf{M} = (M, \cdot, e)$ the **complex algebra** $\mathbf{M}^+$ is the residuated lattice $(\mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{e\})$, where for all $X, Y \subseteq M$,

$$X \cdot Y = \{xy \colon x \in X, y \in Y\},$$

$$X \backslash Y = \{z \in M \colon X \cdot \{z\} \subseteq Y\} \text{ and } X/Y = \{z \in M \colon \{z\} \cdot Y \subseteq X\}.$$

The previous two-component Płonka sum can be used for the 4-element relation algebra $\mathbb{Z}_2^+$ where $\mathbb{Z}_2$ is the 2-element group.



This relation algebra is **not** locally integral, but it is Płonka summable, and the Płonka sum decomposition can be applied to all members of the variety of relation algebras generated by $\mathbb{Z}_2^+$.

How good is this structure theory?

The table below shows how many Płonka summable residuated posets can be built from indecomposable residuated posets (i.e. ones with a unique positive idempotent).

Cardinality $n =$	1	2	3	4	5	6	7	8
Residuated posets (RP)	1	2	5	28	186	1795		
Residuated lattices	1	1	3	20	149	1488	18554	295292
Decomposable RP	1	2	5	24	134	1029		
Płonka summable RP	1	2	5	23	121	889		
Unique pos. idem. RP	1	2	4	16	72	516		
Com. idem. Pł. summ. RP	1	1	2	5	13	36	107	
Idempotent integral RP	1	1	1	2	3	5	8	15

E.g., for commutative idempotent Płonka summable residuated posets, **99** seven-element RPs can be constructed from **13** indecomposable components.

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THANKS!