On equational bases for the benzene ortholattice and Płonka sums of generalized Boolean algebras

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Outline

Part 1: Joint work with **J.B. Nation and Ralph Freese**, U. Hawaii

Ortholattice varieties and some equational bases

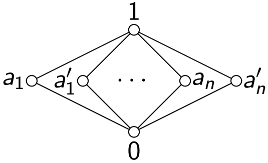
Part 2: Joint work with **Melissa Sugimoto**, U. Leiden

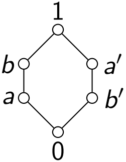
Involutive ℓ-semilattices and Plonka sums of generalized Boolean algebras

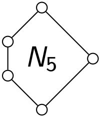
Ortholattices

An **ortholattice** $(A, +, \cdot, ', 0, 1)$ is a lattice $(A, +, \cdot)$ with a unary **orthocomplement** $'$ that satisfies

$$x'' = x, \quad (x + y)' = x' \cdot y', \quad x \cdot x' = 0 \quad \text{and} \quad x + x' = 1.$$

Examples: Boolean algebras, $MO_n =$ 

Benzene hexagon $H =$ 

Not an OL: 

Varieties of ortholattices

A **variety of ortholattices** is a class of all ortholattices that satisfy a given set E of ortholattice identities.

In this case E is an **equational basis** for the variety it defines.

Example: $T = \{x = y\}$ is a basis for all **one-element** ortholattices

$D = \{x(x' + y) + y = y\}$ is an **OL basis** for all Boolean algebras

$M = \{(xz + y)z = xz + yz\}$ is a basis for all **modular** ortholattices

$O = \{x+x'(x+y) = x+y\}$ is a basis for all **orthomodular** lattices

Generating varieties of ortholattices

Any **intersection** of varieties is again a variety.

For a class $\mathcal{K}$ of ortholattices, let $\mathbb{V}(\mathcal{K})$ be the **smallest variety** containing $\mathcal{K}$.

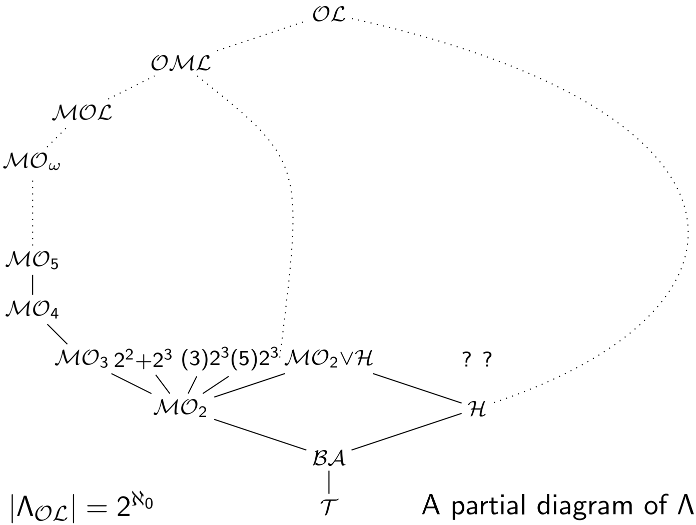
By **Birkhoff's HSP theorem**, $\mathbb{V}(\mathcal{K}) = \mathbb{HSP}(\mathcal{K})$, where
$\mathbb{P}$ = all products, $\mathbb{S}$ = all subalgebras,
$\mathbb{H}$ = all homomorphic images of members of $\mathcal{K}$.

Examples: $\mathcal{MO}_n = \mathbb{V}(MO_n)$

$\mathcal{H} = \mathbb{V}(H)$ the variety generated by the **hexagon benzene ring**.

The set of all ortholattice varieties is a **complete lattice** ordered by inclusion. $\mathcal{V} \wedge \mathcal{W} = \mathcal{V} \cap \mathcal{W}$ and $\mathcal{V} \vee \mathcal{W} = \mathbb{V}(\mathcal{V} \cup \mathcal{W})$

The lattice $\Lambda_{\mathcal{OL}}$ of varieties of ortholattices



$|\Lambda_{\mathcal{OL}}| = 2^{\aleph_0}$

A partial diagram of $\Lambda_{\mathcal{OL}}$

Equational bases for some varieties

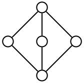
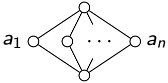
Baker [1972] proved that any **congruence distributive variety** that is generated by a finite algebra has a **finite equational basis**.

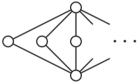
For bounded lattices L, M the (glued) **horizontal sum** $L +_h M$ is the disjoint union with the bounds identified. If L, M are ortholattices, so is $L +_h M$, and the **orthomodular identity is preserved**.

Bruns and Kalmbach [1971] found equational bases for all varieties of orthomodular lattices that are generated by **finite horizontal sums of finite Boolean algebras**.

In particular, $\mathcal{MO}_2$ has a **3-variable equational basis** $c(x, y) + c(x, z) + c(y, z) = 1$, where $c(x, y) = xy + x'y + xy' + x'y'$.

Lattice equational bases for M_n, MO_n

$M_3 =$  $\quad M_n =$ 

$\quad M_\omega =$ 

Jónsson [1968] $\mathcal{M}_\omega$ has basis $E = \{w(x+yz)(y+z) \le x+wy+wz\}$

$\mathcal{MO}_\omega$ has the same **lattice basis** relative to $\mathcal{OL}$.

$\mathcal{M}_n$ has basis $E_n = E \cup \{w \cdot \prod_{1\le i<j\le n}(x_i + x_j) \le wx_1 + wx_2 + \cdots + wx_n\}$

E.g. $\mathcal{M}_3$ has basis $w(x_1+x_2)(x_1+x_3)(x_2+x_3) \le wx_1 + wx_2 + wx_3$

$\mathcal{M}_4$ has a **5-variable basis**, and $\mathcal{MO}_2$ has the **same lattice** basis.

$\mathcal{MO}_n$ has a $2n{+}1$-**variable lattice basis** E_{2n}.

An equational basis for the hexagon variety $\mathcal{H}$?

In Sept 2020 **John Harding** sent me an email about finding an equational basis for $\mathcal{H}$.

Kirby Baker's finite basis theorem is in principle **constructive**, but in practice not feasible even for very small algebras.

Roberto Giuntini proposed a 3-variable basis

$$B = \{(x+y)(x+z)(x'+yz) = (x+yz)(x'+yz),$$
$$(x+y)(x'+y)+xy' = x+y\}$$

McKenzie [1972] found a 4-variable basis for the lattice variety $\mathcal{N}_5$

$$M = \{w(x+y)(x+z) \leq w(x+yz)+wy+wz,$$
$$w(x+y(w+z)) = w(x+wy)+w(wx+yz)\}$$

We also investigated whether this is a basis for $\mathcal{H}$, but (at that time) no progress after a few weeks.

When is an OL variety defined by lattice equations?

Joint work with **J.B. Nation and Ralph Freese** (Jan 2022).

RdK denotes the **lattice reduct** of an ortholattice K.

Let $\Lambda_{\mathcal{L}}$ be the lattice of varieties of lattices and define $\rho : \Lambda_{\mathcal{OL}} \to \Lambda_{\mathcal{L}}$ by $\rho(\mathcal{V}) = \mathbb{V}(\{\mathrm{Rd}K \mid K \in \mathcal{V}\})$.

(i) Describe the range of ρ.

(ii) When is a variety $\mathcal{V}$ of ortholattices determined by an equational basis of $\rho(\mathcal{V})$?

Note: Varieties in the range of ρ are **self-dual**.

If k is odd then $\mathbb{V}(M_k)$ is **not** in the range of ρ.

An embedding $h : L \hookrightarrow \prod L_i$ is **subdirect** if $(\pi_i \circ h)[L] = L_i$ for all $i \in I$

L is **subdirectly irreducible** if $L \overset{sd}{\hookrightarrow} \prod L_i$ implies $L \cong L_i$ for some $i \in I$

Theorem

Let L be a finite s.i. lattice. Then L is a lattice-subdirect factor of an ortholattice if and only if there exists an ortholattice S such that $\mathrm{Rd}S \overset{sd}{\hookrightarrow} L \times L^d$, where L^d is the dual of L.

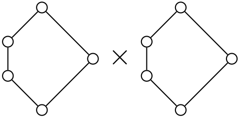
Proof (outline).

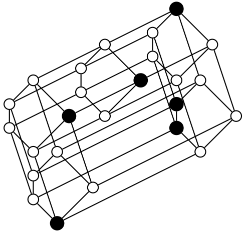
Let $K \in \mathcal{OL}$ and θ a lattice congruence with $(\mathrm{Rd}K)/\theta \cong L$.
On K define θ' by $x\theta' y \iff x'\theta y'$.
Then θ' is a lattice congruence (by De Morgan's law),
$(\mathrm{Rd}K)/\theta' \cong L^d$ and $\theta\cap\theta'$ is an ortholattice congruence
(since $x\theta\cap\theta' y \iff x'\theta'\cap\theta y'$). So take $S = K/\theta\cap\theta'$, then
$\mathrm{Rd}S \overset{sd}{\hookrightarrow} \mathrm{Rd}K/\theta \times \mathrm{Rd}K/\theta' \cong L \times L^d$. □

Deciding if $\mathbb{V}(L \times L^d)$ is in the range of ρ

For a finite s.i. lattice L, check if there exists a **subdirectly** embedded sublattice S of $L \times L^d$ that supports an orthocomplement.

Example: $\mathbb{V}(N_5 \times N_5^d) = \mathbb{V}(N_5) = \rho(\mathbb{V}(H))$ since $H \overset{sd}{\hookrightarrow} N_5 \times N_5^d$.





Any lattice basis for $\mathbb{V}(N_5)$ is a basis for $\mathbb{V}(H)$

Let K be an ortholattice such that $\mathrm{Rd}K \in \mathbb{V}(N_5)$.

Then $\mathrm{Rd}K$ has a subdirect embedding into a product of copies of N_5 and 2.

As in the proof of the preceding theorem, every N_5-congruence $\theta \in \mathrm{Con}(\mathrm{Rd}K)$ is paired with $\theta' = \{(x, y) \mid x'\theta y'\}$, and $\bar{\theta} := \theta \cap \theta'$ is an ortholattice congruence.

Thus we get an embedding of K into a product of $K/\bar{\theta}$ and copies of 2, where θ ranges over all N_5-congruences.

Since $K/\bar{\theta}$ is an orthocomplemented sublattice of $N_5 \times N_5$, it suffices to check that all subdirect sublattices of $N_5 \times N_5$ that admit an orthocomplement are isomorphic to H.

Any lattice basis for $\mathbb{V}(N(L))$ is a basis for $\mathbb{V}(L+L^d)$

This was first checked with a computer calculation for $N_5 \times N_5$.

Later generalized by hand to cover all lattices $N(L) = L + \{c\}$ where L is a finite subdirectly irreducible lattice.

(For lattices L, M the (loose) **parallel sum** $L + M$ is the disjoint union of L and M with a **new** $0, 1$ added.)

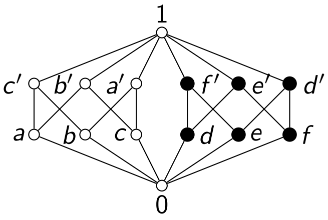
Note: $L + L^d$ is orthocomplemented by the map $x \leftrightarrow x^d$, $0 \leftrightarrow 1$.

Theorem

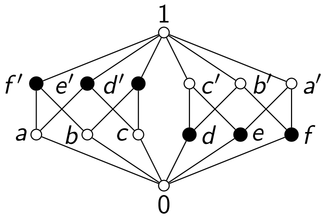
For any finite subdirectly irreducible lattice L, the ortholattice variety $\mathbb{V}(L + L^d)$ is determined by lattice identities.

$\mathcal{H}$ is covered by the case when $L = 2$.

Lattices with several (nonisomorphic) orthocomplements



$2^3 +_h 2^3$

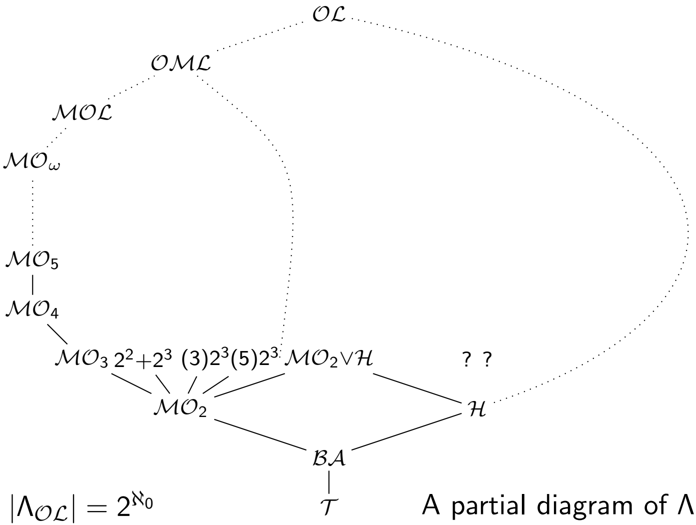


K

These two ortholattices cannot be distinguished by lattice identities.

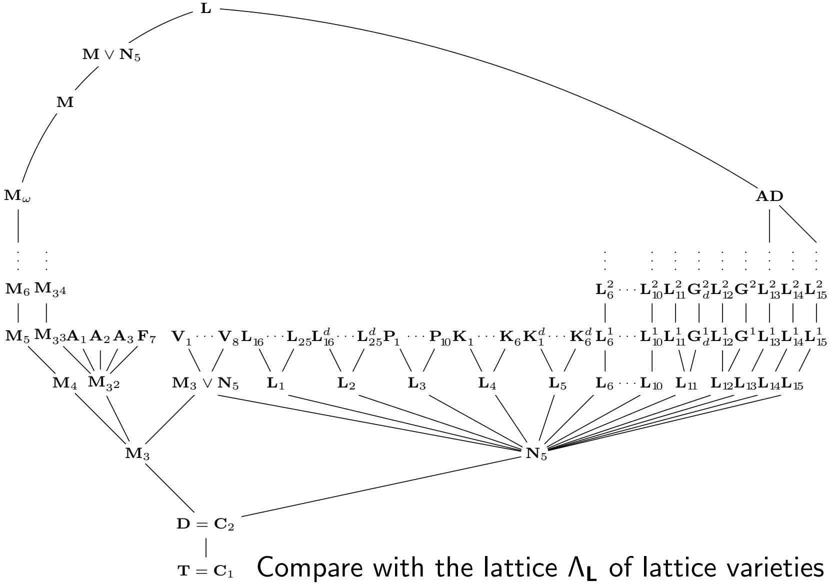
However $2^3 +_h 2^3$ is orthomodular, whereas H is a subalgebra of K.

Recall: $\Lambda_{\mathcal{OL}}$ lattice of ortholattice varieties



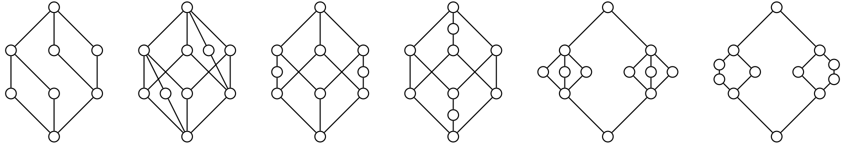
$|\Lambda_{\mathcal{OL}}| = 2^{\aleph_0}$

A partial diagram of $\Lambda_{\mathcal{OL}}$

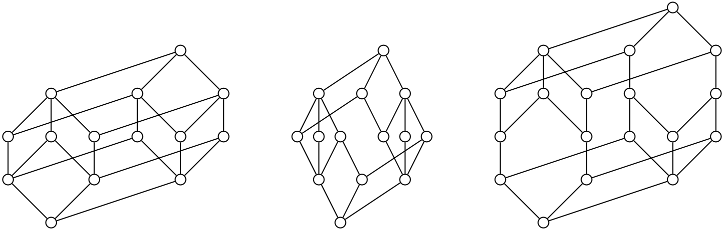


Compare with the lattice $\Lambda_{\mathbf{L}}$ of lattice varieties

Nine ortholattices that generate covers of $\mathbb{V}(H)$



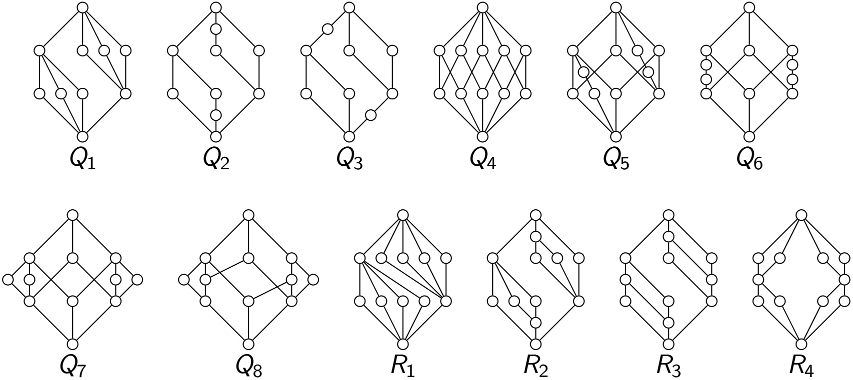
O_1 O_2 O_3 O_4 O_5 O_6



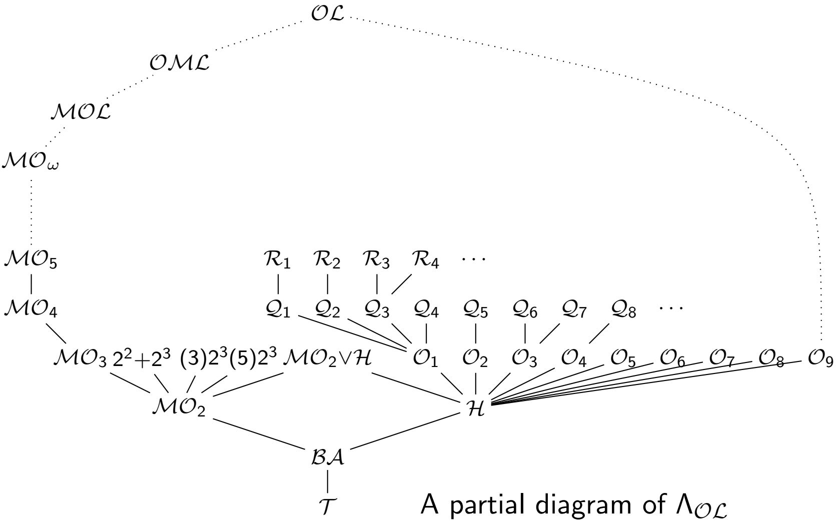
O_7 O_8 O_9

$\mathcal{O}_5$ shows that a basis for $\mathbb{V}(H)$ requires 3 variables.

Other subdirectly irreducible ortholattices



More details of the lattice $\Lambda_{\mathcal{OL}}$ of ortholattice varieties



A partial diagram of $\Lambda_{\mathcal{OL}}$

A full list of covering varieties gives a test for bases

Suppose $\mathcal{V}$ is a variety and $\mathcal{C}$ is a collection of varieties that **strongly cover** $\mathcal{V}$, i.e. for all varieties $\mathcal{W}$, $\mathcal{V} \subset \mathcal{W}$ implies $\mathcal{U} \subseteq \mathcal{W}$ for some $\mathcal{U} \in \mathcal{C}$.

Then E is a basis for $\mathcal{V}$ **iff** $\mathcal{V} \models E$ and for all $\mathcal{U} \in \mathcal{C}$, $\mathcal{U} \not\models E$.

Jónsson and Rival [1979] $\mathcal{M}_3 \vee \mathcal{N}_5, \mathbb{V}(L_1), \ldots, \mathbb{V}(L_{15})$ strongly cover $\mathcal{N}_5$. ($L_1, \ldots, L_{15}$ were found by **McKenzie** [1972].)

$\Rightarrow$ can easily test lattice identities to see if they are a basis for $\mathcal{N}_5$.

If so, then by the **preceding results** they are also a basis for $\mathcal{H}$.

But to test ortholattice identities we need a full list of covers of $\mathcal{H}$

Is $\mathcal{MO}_2 \vee \mathcal{H}, \mathcal{O}_1, ..., \mathcal{O}_9$ a full list of covers of $\mathcal{H}$?

So far we have proved the following result.

Theorem

If a finite ortholattice K has an atom a such that $\downarrow a'$ is not a prime ideal, then there exists $x \in K$ such that $Sg(a, x)$ contains MO_2 or O_j for some $j \in \{1, 2, 3, 4, 8\}$.

Now can assume that K is a finite ortholattice in which $\downarrow a'$ is a prime ideal for every atom a. If $K \notin \mathcal{H}$ then show K contains MO_2 or O_j for some $j \in \{4, 5, 6, 7, 9\}$.

Last step would be to remove finiteness of K.

If $\mathcal{MO}_2 \vee \mathcal{H}, \mathcal{O}_1, ..., \mathcal{O}_9$ is a full list of covers of $\mathcal{H}$ then **Roberto Giuntini's identities B are also a basis for $\mathcal{H}$.**

Some references for Part 1

- K. Baker, Finite equational bases for finite algebras in a congruence-distributive equational class, Advances in Math. 24 (1977), 207–243.
- G. Bruns and G. Kalmbach: Varieties of orthomodular lattices, Canadian J. Math., Vol. XXIII, No. 5, 1971, pp. 802–810
- B. Jónsson: Equational classes of lattices. Math. Scand., 22:187–196, 1968.
- B. Jónsson and I. Rival: Lattice varieties covering the smallest nonmodular variety. Pacific J. Math., 82(2):463–478, 1979.
- R. McKenzie: Equational bases and nonmodular lattice varieties. Trans. Amer. Math. Soc., 174:143, 1972.

Part 2

Joint work with **Melissa Sugimoto**, U. Leiden

Involutive ℓ-semilattices and Plonka sums of generalized BAs

Involutive po-semigroups

An **involutive po-semigroup** or **ipo-semigroup** $(A, \leq, \cdot, \sim, -)$ is a poset $(A, \leq)$ with an associative binary operation $\cdot$, two unary **order-reversing operations** $\sim, -$ that are an **involutive pair**: ${\sim}{-}x = x = {-}{\sim}x$, and for all $x, y, z \in A$

$$\text{(ires)} \quad xy \leq z \iff x \leq -(y \cdot {\sim}z) \iff y \leq {\sim}(-z \cdot x).$$

It follows that ipo-semigroups are **residuated**.

Hence $\cdot$ is order-preserving.

A convenient **equivalent** formulation of (ires):

$$\text{(rotate)} \quad xy \leq z \iff y \cdot {\sim}z \leq {\sim}x \iff -z \cdot x \leq -y.$$

Involutive po-monoids

A **ipo-monoid** $(A, \leq, \cdot, 1, \sim, -)$ is an ipo-semigroup $(A, \leq, \cdot, \sim, -)$ such that $1x = x = x1$.

In this case we denote -1 by 0 and (rotate) can be replaced by

$$x \leq y \iff x \cdot {\sim}y \leq 0 \iff -y \cdot x \leq 0.$$

Note that ${\sim}1 = 0$, $1 = -0 = {\sim}0$.

The class of ipo-monoids includes **all groups** (if $\leq$ is $=$) and

all partially ordered groups where ${\sim}x = -x = x^{-1}$.

MV-algebras are ipo-monoids, in fact **iℓ-monoids** ($\vee, \wedge$ are definable)

Involutive po-semilattices

An **ipo-semilattice** $(A, \leq, \cdot, -)$ is an ipo-semigroup where $\cdot$ is commutative and idempotent. (Commutativity implies $\sim x = -x$.)

In an ipo-semilattice there is another partial order $\sqsubseteq$ called the **multiplicative order**, defined by $x \sqsubseteq y \iff xy = x$.

Examples of ipo-semilattices: Boolean algebras $(A, \leq, \cdot, -)$, where join is $-(-x \cdot -y)$.

They form a **po-subvariety** defined by $x \cdot -x \leq y \cdot -y$.

More generally, ipo-semilattices can be visualized by the **two Hasse diagrams for $\leq$, $\sqsubseteq$**

Visualizing ipo-semilattices

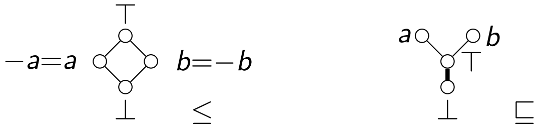


Figure: Partial order and multiplicative order of the smallest ipo-semilattice that does not have an identity element.

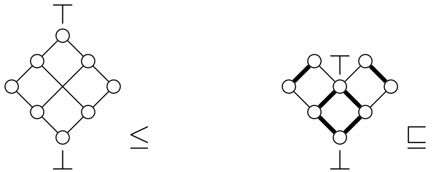


Figure: Smallest ipo-semilattice that is not lattice-ordered.

Unital involutive po-semilattices

An element t in an ipo-semilattice is the **multiplicative identity** iff t is the top element in the multiplicative order.

Hence an ipo-semilattice is **unital** if and only if the multiplicative order has a **top element**.

Sugihara monoid reducts without $\wedge, \vee$ are unital ipo-semilattices.

For finite commutative idempotent involutive residuated lattices (CIdInRL for short) a **full structural description** has been given by **[J., Tuyt, Valota 2021]**.

An **iℓ-semigroup** $(A, \vee, \cdot, \sim, -)$ is an ipo-semigroup where the **poset is a lattice** and $\vee$ (hence $\wedge$) are part of the signature.

Structural description for ipo-semilattices

We give a description of finite ipo-semilattices based on **Płonka sums of generalized Boolean algebras**.

Similar methods are used by Jenei [2022] to describe the structure of **even and odd involutive commutative residuated chains**.

Inspired by a duality for involutive bisemilattices by Bonzio, Loi, Peruzzi [2019], we give a more compact dual description of finite ipo-semilattices based on **semilattice direct systems of partial maps between sets**.

Lemma 1

Let A be a **residuated po-semilattice** and let $x, y \in A$ such that $x\backslash x = y\backslash y$. Then

1. $x \sqsubseteq y \iff x \leq y$,
2. $x\backslash x = xy\backslash xy$,
3. if $y\backslash y = z\backslash z$ then $x\backslash x = yz\backslash yz$, and
4. if $y \sqsubseteq z \sqsubseteq x\backslash x$ then $x\backslash x = z\backslash z$.

Defining an Equivalence Relation

Define an equivalence relation $\equiv$ on A by $x \equiv y \iff x\backslash x = y\backslash y$. Part (1) of the previous lemma shows that the partial order $\leq$ and the semilattice order $\sqsubseteq$ **agree on each equivalence class** of $\equiv$.

The term $x\backslash x$ is denoted by 1_x.

Lemma 2

Let A be an rpo-semilattice and define $\equiv$ as above. Then each equivalence class of $\equiv$ **is a semilattice** $([x]_\equiv, \cdot)$ with identity element 1_x.

Note: In an ipo-semilattice $1_x = x\backslash x = -(x \cdot -x)$.

In an ipo-semilattice define $0_x = -1_x$ or equivalently $0_x = x \cdot -x$.

Lemma 3

Let A be an ipo-semilattice and define

$$\mathbb{B}_x = \{a \in A \mid 0_x \sqsubseteq a \sqsubseteq 1_x\}.$$

Then

1. the intervals $\mathbb{B}_x$ are closed under negation, i.e., $y \in \mathbb{B}_x \implies -y \in \mathbb{B}_x$,
2. $x \sqsubseteq y$ implies $0_x \sqsubseteq 0_y$ and $0_x \leq 0_y$,
3. $0_x \sqsubseteq 0_y$ if and only if $0_x \leq 0_y$,
4. $x \sqsubseteq y$ implies $1_x \sqsubseteq 1_y$, and
5. $1_x \cdot 1_y = 1_{xy}$.

ipo-semilattices are unions of Boolean algebras

Define $x + y = -(-y \cdot -x)$.

Theorem 1. Partition by Boolean Algebras

Given an ipo-semilattice A, the semilattice intervals $(\mathbb{B}_x, \cdot, +, -, 0_x, 1_x)$ are Boolean algebras and they **partition** A.

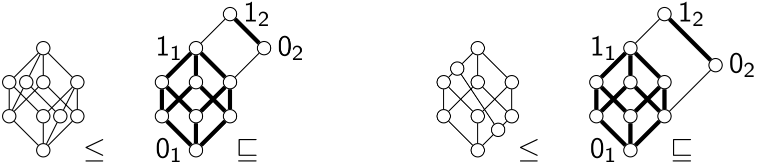
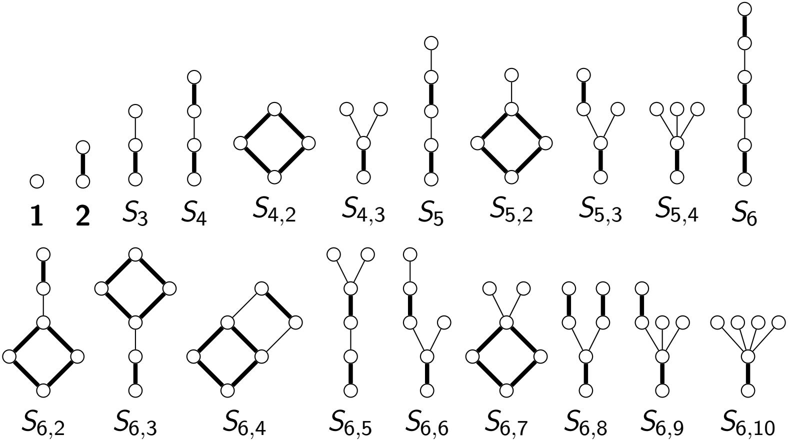


Figure: (Right) unital ipo-semilattice that is **not** an iℓ-semilattice.

The Boolean components are denoted by **thick lines** and are connected by homomorphisms (thin lines). For **CldInRL** the above theorem is due to **[J., Tuyt, Valota 2021]**.

Note: A finite iℓ-semilattice is a (nonunital) **commutative idempotent involutive (i.e. Frobenius) quantale**.

Now we can construct all these algebras (only $\sqsubseteq$ is shown):



Subdirectly irreducible unital iℓ-semilattices

Lemma

Let **A** be a unital ipo-semilattice. If $0_x = 1_x$ then $x = 1$, hence all Boolean components except possibly the top one are nontrivial.

A unital ipo-semilattice is called **odd** if it satisfies the identity $-1 = 1$ (i.e., $0 = 1$).

Theorem 2.

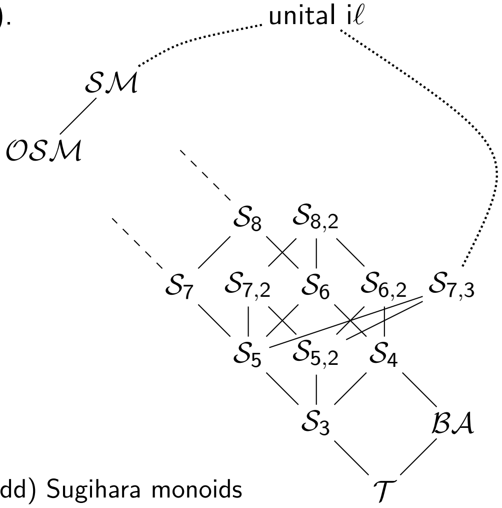
A finite unital ipo-semilattice **A** is odd if and only if $|A|$ is odd.

A finite unital iℓ-semilattice **A** is subdirectly irreducible if and only if 1 has a unique coatom in the monoidal order.

# of elem. $n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
unital iℓ-semilats	1	1	1	2	2	4	4	9	10	21	22	49	52	114	121	270
subdir. irreducible	0	1	1	1	2	2	4	4	9	10	21	22	49	52	114	121

Some join-irreducible subvarieties of unital iℓ-semilattices

Let $\mathcal{S}_{i,j} = \mathsf{Var}(S_{i,j})$.



$(\mathcal{O})\mathcal{SM}$ = (odd) Sugihara monoids

Some equational bases

The previous diagram is complete below $\mathcal{SM}$ and $\mathcal{S}_{5,2}$.

Hence we have full lists of covering varieties for proper subvarieties of $\mathcal{SM}$ (excluding $\mathcal{OSM}$).

$\mathcal{BA}$ is covered only by $\mathcal{S}_3 \vee \mathcal{BA}$, so $x0 = 0$ is a basis relative to $\mathcal{SM}$

$\mathcal{S}_3$ has $(x \vee -x)(0 \vee -y) = x \vee -(xy)$ as basis relative to $\mathcal{OSM}$.

$\mathcal{S}_4$ has $0 \leq x \vee -(xy)$ as basis relative to $\mathcal{SM}$.

$\mathcal{S}_{5,2}$ has $(x \vee -x)(0 \vee -y) = x \vee -(xy)$ as basis relative to odd unital iℓ-semilattices.

Theorem 3.

Let **A** be an iℓ-semilattice. Then for every $x \in A$ the multiplicative downset of 1_x is a **unital** sub-iℓ-semilattice.

Proof

- Let $\mathbf{A}_x$ denote the multiplicative downset of 1_x. If $y \cdot 1_x = y$ and $z \cdot 1_x = z$ then $(y \vee z) \cdot 1_x = (y \cdot 1_x) \vee (z \cdot 1_x) = y \vee z$ since $\cdot$ distributes over $\vee$. Therefore $\mathbf{A}_x$ is closed under join.
- Each Boolean component is closed under $-$, so it is clear that $\mathbf{A}_x$ is closed under $-$.
- By DeMorgan laws, closure under $-$ and $\vee$ guarantees closure under $\wedge$.

Therefore $\mathbf{A}_x$ is a sub-iℓ-semilattice.

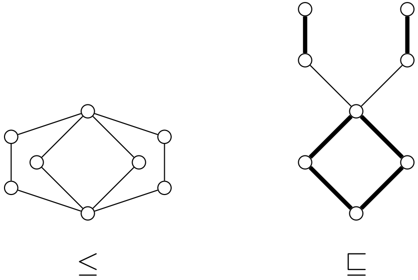


Figure: An 8-element iℓ-semilattice. Its multiplicative order shows its unital sub-iℓ-semilattices.

Semilattice direct systems and Płonka sums

A **semilattice direct system** (or **sd-system** for short) is a triple $\mathbf{B} = (\mathbf{B}_i, h_{ij}, I)$ such that

- I is a semilattice,
- $\{\mathbf{B}_i : i \in I\}$ is a family of **algebras of the same type** with **disjoint** universes,
- $h_{ij} : \mathbf{B}_i \to \mathbf{B}_j$ is a homomorphism for all $i \geq j \in I$ such that h_{ii} is the identity on $\mathbf{B}_i$ and for all $i \geq j \geq k$, $h_{jk} \circ h_{ij} = h_{ik}$.

The **Płonka sum** over $\mathbf{B}$ is the algebra $P_{ł}(\mathbf{B}) = \bigcup_{i \in I} \mathbf{B}_i$ with each fundamental operation $g^{\mathbf{B}}$ defined by

$$g^{\mathbf{B}}(b_{i_1}, \ldots, b_{i_n}) = g^{\mathbf{B}_j}(h_{i_1 j}(b_{i_1}), \ldots, h_{i_n j}(b_{i_n}))$$

where $b_{i_k} \in \mathbf{B}_{i_k}$ and $j = i_1 \cdots i_n$ is the semilattice meet of $i_1, \ldots, i_n \in I$.

iℓ-semilattices are multiplicative Płonka sums

Theorem 4.

Let $\mathbf{A}$ be an iℓ-semilattice, and define $I = (\{1_x \mid x \in A\}, \cdot)$. Then

1. $\mathbf{B} = (\mathbb{B}_i, h_{ij}, I)$ is a sd-system of Boolean algebras, where each $h_{ij} : \mathbb{B}_i \to \mathbb{B}_j$ is a generalized Boolean algebra homomorphism (i.e., mapping 1_i to 1_j but not 0_i to 0_j) defined by $h_{ij}(x) = x \cdot j$,
2. the image $h_{ij}[\mathbb{B}_i]$ is a proper filter,
3. the Płonka sum $P_{\nmid}(\mathbf{B})$ reconstructs the reduct algebra $(A, \cdot, -)$.

Reconstructing the lattice order takes more work.

Colimits of finite unital sub-iℓ-semilattices

Theorem 5.

Let $\mathbf{A}$ be a finite iℓ-semilattice, define $I = (\{1_x \mid x \in A\}, \cdot)$ and let A_i be the multiplicative downset of $i \in I$.
Then $\{A_i : i \in I\}$ is a system of finite unital subalgebras of A such that $A_i \cap A_j = A_{ij}$ and $A = \sum_{i \in I} A_i$.

By [J., Tuyt, Valota 2021] each finite unital iℓ-semilattice is determined by its monoidal semilattice, so the above theorem extends this result to nonunital iℓ-semilattices.

The same result is conjectured to hold for ipo-semilattices.

Dual Representation by Partial Functions Between Sets

Partial Functions

Definition. A **proper partial function** $f : X \to Y$ is a function from U to Y where $U \subsetneq X$ is the domain of f.

Developing a Dual Representation

Given an ipo-semilattice $\mathbf{A}$, it is a partition of Boolean components by Theorem 1.

Each Boolean component is determined by its set of atoms.

The partial functions map between sets of atoms (opposite to homomorphisms).

A dual representation of sd-systems of Boolean algebras gives a much more compact way of drawing finite ipo-semilattices.

Dual Representation by Partial Functions Between Sets

Every finite Boolean algebra $\mathbb{B}_i$ is **isomorphic** to the powerset Boolean algebra of its finite set X_i of atoms.

For $i \le j$, the **generalized BA homomorphism** h_{ji} corresponds to the **partial map** $f_{ij} : X_i \to X_j$ defined by

$$f_{ij}(a) = b \iff a \le h_{ji}(b) \text{ and } a \not\le h_{ji}(0_j).$$

A **sd-system of proper partial maps** is a triple $\mathbf{X} = (X_i, f_{ij}, I)$ such that

- I is a semilattice,
- $\{X_i : i \in I\}$ is a family of disjoint sets, and
- $f_{ij} : X_i \to X_j$ is a proper partial map for all $i \le j \in I$ such that $f_{ii} = id_{X_i}$ and for all $i \le j \le k$, $f_{jk} \circ f_{ij} = f_{ik}$.

Dual Representation by Partial Functions Between Sets

Lemma

In every ipo-semilattice $x, y \sqsubseteq z \implies 0_x \cdot 0_y = 0_{xy}$.

An sd-system of partial maps is **covering** if for all $i, j \leq k$ with $i \cdot j = \ell$, $\mathrm{dom}(f_{\ell,i}) \cup \mathrm{dom}(f_{\ell,j}) = X_\ell$.

Corollary

Every sd-system of partial maps of an ipo-semilattice is covering.

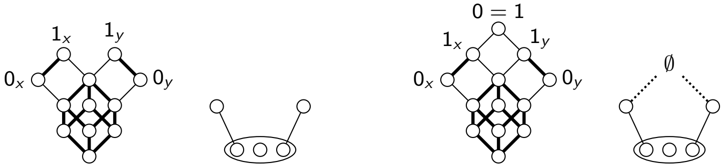


Figure: A nonunital ipo-semilattice that has no unital completion

Beyond idempotence and commutativity

Current joint work with Sid Lodhia and José Gil-Ferez

- For a suitable subvariety of involutive residuated lattices, the finite members are disjoint unions of MV-algebras.
- This uses a Płonka sum with generalized MV-algebra homomorphisms.
- The dual representation of partial functions between sets generalizes to a dual representation of partial functions between multisets.
- In a more general setting, a large class of involutive residuated lattices can be constructed from disjoint unions of integral involutive residuated lattices.

Some references for Part 2

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THANKS!