

# Nonassociative right hoops

**Peter Jipsen**, Chapman University

joint work with **Michael Kinyon**, University of Denver

Workshop on Algebraic Proof Theory for Social Behavior

University of Delft

July 4, 2018

# Overview

- Right residuated posets
- Nonassociative right hoops
- Equational axioms
- Independence of axioms
- Unital nonassociative right hoops
- Characterizing unital congruences

# Introduction

A **residuated magma** is a partially ordered algebra  $(A, \leq, \cdot, /, \backslash)$  such that

- $(A, \leq)$  is a poset,
- $\cdot$  is a binary operation and
- $/, \backslash$  are the right and left residuals of  $\cdot$ , i.e., the residuation property

$$x \cdot y \leq z \quad \iff \quad x \leq z / y \quad \iff \quad y \leq x \backslash z$$

holds for all  $x, y, z \in A$ .

As usual, we abbreviate  $x \cdot y$  by  $xy$  and adopt the convention that  $\cdot$  binds stronger than  $/, \backslash$ .

If the operation  $\backslash$  is omitted then the algebra  $(A, \leq, \cdot, /)$  is called a **right-residuated magma**.

## Introduction

Define the term  $x \wedge y = (x/y)y$  and consider the following two varieties:

- A **right quasigroup** is an algebra  $(A, \cdot, /)$  satisfying the identities

$$x \wedge y = x = (xy)/y.$$

Right quasigroups are precisely those right-residuated magmas for which the partial order  $\leq$  is the equality relation.

- A **right hoop** is an algebra  $(A, \cdot, /)$  satisfying the identities

$$x \wedge y = y \wedge x, (x/x)y = y \text{ and } x/(yz) = (x/z)/y.$$

Then it turns out that  $x/x$  is a constant (denoted by 1),  
the operation  $\cdot$  is associative,  
the operation  $\wedge$  is a semilattice operation,  
1 is the top element with respect to the semilattice order  $\leq$ , and  
 $/$  is the right residual of  $\cdot$  with respect to  $\leq$ .

# Introduction

Right hoops were introduced by **Bosbach** [1969,1970] under the name “left complementary semigroups”

**Büchi and Owens** [1975] studied the case where  $\cdot$  is commutative, referring to these structures as “hoops”.

The partial order is definable in both cases, which motivates the next definition.

## Nonassociative right hoops

A **nonassociative right hoop**  $(A, \leq, \cdot, /)$ , or **narhoop** for short, is a right-residuated magma such that for all  $x, y \in A$

- (N)  $x \leq y \iff x \wedge y = x = y \wedge x.$

In any right-residuated magma  $(x/y)y \leq x$  or equivalently  $x \wedge y \leq x$  holds for all  $x, y$ ,

hence in a narhoop (N) implies that the identity  $(x \wedge y) \wedge x = x \wedge y$  holds.

This provides an alternative definition for narhoops:

they are right-residuated magmas that satisfy the identity

- (N1)  $(x \wedge y) \wedge x = x \wedge y$  and

- (N')  $x \leq y \iff x = y \wedge x$

since in the presence of (N1), if  $x = y \wedge x$  then multiplying by  $y$  on the right we have  $x \wedge y = (y \wedge x) \wedge y = y \wedge x = x.$

## Nonassociative (left) hoops

A **nonassociative left hoop** or **nalhoop**  $(A, \leq, \cdot, \backslash)$  is defined dually

A **nonassociative hoop** or **nahoop** is both a narhoop and a nalhoop

We consider only narhoops in this talk.

The two motivating varieties fit into this framework as follows.

- A narhoop  $(A, \leq, \cdot, /)$  is a right quasigroup **if and only if**  $\leq$  is the equality relation.
- A narhoop  $(A, \leq, \cdot, /)$  is a right hoop **if and only if**  $x/yz = (x/z)/y$  and the quasiequation  $x \wedge y = x \Rightarrow x \leq y$  holds.

## Main result

**Narhoops form a finitely based variety of algebras.**

We assume that  $x/y$  binds stronger than  $x \wedge y = (x/y)y$ .

### Theorem

*Let  $(A, \leq, \cdot, /)$  be a narhoop. Then the following identities hold:*

$$(N1) \quad (x \wedge y) \wedge x = x \wedge y$$

$$(N2) \quad xy/y \wedge x = x$$

$$(N3) \quad xz \wedge (x \wedge y)z = (x \wedge y)z$$

$$(N4) \quad (x/z) \wedge (x \wedge y)/z = (x \wedge y)/z.$$

*Conversely, let  $(A, \cdot, /)$  be an algebra with two binary operations satisfying (N1)–(N4), and define  $x \leq y \iff x = y \wedge x$ . Then the identities*

$$(N5) \quad x \wedge xy/y = x$$

$$(N6) \quad (x \wedge y)/y = x/y$$

$$(N7) \quad (x \wedge y) \wedge y = x \wedge y$$

*hold and  $(A, \leq, \cdot, /)$  is a narhoop.*



## Proof.

Assume  $(A, \leq, \cdot, /)$  is a narhoop.

As noted before, the identity (N1:  $(x \wedge y) \wedge x = x \wedge y$ ) holds in narhoops

Right-residuated magmas also satisfy  $x \leq xy/y$

hence (N2)  $xy/y \wedge x = x$  follows from (N':  $x \leq y \iff x = y \wedge x$ )

Having a right residual implies that right-multiplication is order preserving so  $(x \wedge y)z \leq xz$  holds in all narhoops, which produces (N3).

Similarly the right residual is order preserving in the first argument, hence  $(x \wedge y)/z \leq x/z$  holds, and now (N4) follows from (N').

## Proof continued.

Conversely, suppose  $(A, \cdot, /)$  satisfies (N1)–(N4), and  $\leq$  is defined by (N')

From (N2:  $xy/y \wedge x = x$ ), (N1:  $(x \wedge y) \wedge x = x \wedge y$ ) and (N2) again, we get (N5):

$$x \wedge (xy/y) = (xy/y \wedge x) \wedge (xy/y) = (xy/y) \wedge x = x$$

Replace  $x$  in (N5) by  $x/y$  to get  $x/y \wedge (x \wedge y)/y = x/y$

then use (N4:  $(x/z) \wedge (x \wedge y)/z = (x \wedge y)/z$ ) to get N6:  $(x \wedge y)/y = x/y$

To prove (N7:  $(x \wedge y) \wedge y = x \wedge y$ ) multiply (N6) on the right by  $y$

## Proof continued.

Now reflexivity of  $\leq$  follows from (N5) and (N1):

$$x \wedge x = (x \wedge xy/y) \wedge x = x \wedge (xy/y) = x.$$

For antisymmetry, if  $x \leq y$  and  $y \leq x$ , then  $x \wedge y = x = y \wedge x$  and  $y = x \wedge y$ , hence  $x = y$ .

Transitivity: Suppose  $x \leq y$  and  $y \leq z$  so that  $x \wedge y = x = y \wedge x$  and  $y \wedge z = y = z \wedge y$ . First, note that

$$z/x \wedge y/x = z/x \wedge (z \wedge y)/x = (z \wedge y)/x = y/x$$

using (N4) in the second equality.

## Proof continued.

Now we compute

$$\begin{aligned} z \wedge x &= (z \wedge x) \wedge x && \text{by N6: } (x \wedge y)/y = x/y \\ &= (z \wedge x) \wedge (y \wedge x) = (z/x)x \wedge (y/x)x \\ &= (z/x)x \wedge (z/x \wedge y/x)x && \text{since } z/x \wedge y/x = y/x \\ &= (z/x \wedge y/x)x && \text{by (N3)} \\ &= (z/x \wedge (z \wedge y)/x)x = ((z \wedge y)/x)x && \text{by (N4)} \\ &= (z \wedge y) \wedge x = y \wedge x = x. \end{aligned}$$

From  $x = z \wedge x$  we deduce  $x \wedge z = (z \wedge x) \wedge z = z \wedge x$  by (N1), hence  $x \leq z$ .

## Proof continued.

Finally, we prove  $/$  is the right residual of  $\cdot$  with respect to  $\leq$

The right residuation property is equivalent to  $(x/y)y \leq x \leq xy/y$  and  $x \leq y$  implies  $xz \leq yz$  and  $x/z \leq y/z$

Note that (N2) and (N') show  $x \leq xy/y$ .

If  $x \leq y$ , then (N3) gives

$$yz \wedge xz = yz \wedge (y \wedge x)z = (y \wedge x)z = xz,$$

and so (N') implies  $xz \leq yz$ .

By the same argument, (N4) gives  $x/z \leq y/z$ .

## Proof continued.

To prove  $(x/y)y \leq x$ , or equivalently  $x \wedge y \leq x$ ,

substitute  $x/x$  for  $x$ ,  $x$  for  $y$ , and  $(x \wedge y)/x$  for  $z$  in (N3) to get

$$(x/x)x \wedge (x/x \wedge (x \wedge y)/x)x = (x/x \wedge (x \wedge y)/x)x$$

Using (N4) this simplifies to  $(x \wedge x) \wedge ((x \wedge y)/x)x = ((x \wedge y)/x)x$

so by (N1), (N') and reflexivity we have  $x \wedge y \leq x \wedge x = x$

## Independence of axioms

The equational basis (N1)–(N4) for narhoops is independent as can be seen from algebras  $A_i = \{0, 1\}$  ( $i = 1, 2, 3, 4$ ) that each satisfy the axioms except for (Ni).

- In  $A_1$ ,  $\cdot$  is ordinary multiplication and  $x/y = y$ .
- In  $A_2$ ,  $x \cdot y = x$  and  $x/y = 1$ .
- In  $A_3$ ,  $x \cdot y$  is addition modulo 2 and  $x/y = 0$  except that  $1/0 = 1$ .
- In  $A_4$ ,  $x \cdot y$  is the max operation and  $x/y$  is addition modulo 2.

In general, neither  $\cdot$  nor the term operation  $\wedge$  of a narhoop is associative.

However  $\wedge$  is associative both in right quasigroups and in right hoops.

In right quasigroups, this follows from the identity  $x \wedge y = x$ .

In right hoops,  $\wedge$  turns out to be a semilattice operation (J. 2017, Lem. 4).

In both cases the reduct  $(A, \wedge)$  is a *left normal band*, i.e., an idempotent semigroup satisfying the identity  $x \wedge y \wedge z = x \wedge z \wedge y$ .



If  $(A, \cdot, /)$  is a narhoop and  $B \subseteq A$  is closed under  $\wedge$ , then  $B$  inherits the order  $\leq$  from  $A$ .

## Theorem

Let  $(A, \cdot, /)$  be a narhoop and let  $B \subseteq A$  be closed under  $\wedge$ . The following are equivalent.

- 1  $(B, \wedge)$  is a right normal band;
- 2  $(B, \wedge)$  is a semigroup;
- 3 For all  $x, y \in B$ ,  $x \wedge (y \wedge x) = x \wedge y$

$\wedge$ -reducts of right hoops are semilattices

A nonassociative generalization of right hoops is the following variety of narhoops:

### Theorem

*Let  $(A, \cdot, /)$  be a narhoop and let  $B \subseteq A$  be closed under  $\wedge$ . The following are equivalent.*

- 1  $(B, \wedge)$  is commutative;
- 2 For all  $x, y \in B$ ,  $x \wedge (y \wedge x) = y \wedge x$ .

*When these equivalent conditions hold,  $(B, \wedge)$  is a semilattice.*

In a left normal band, the identity  $x \wedge y \wedge z = x \wedge z \wedge y$  expresses the fact that every downset  $(a] = \{x \in A \mid x \leq a\} = \{a \wedge x \mid x \in A\}$  is a subsemilattice.

The same role is played by  $(x \wedge y) \wedge z = (x \wedge z) \wedge y$  in narhoops.

## Theorem

*Let  $(A, \cdot, /)$  be a narhoop and fix  $a \in A$ . Then the downset  $(a]$  is closed under  $\wedge$  and is a semilattice.*

# Narhoops with a left identity element

## Lemma

Let  $(A, \leq, \cdot, /)$  be a right-residuated magma such that  $x \leq y \iff x = y \wedge x$  holds for all  $x, y \in A$ . Then

- 1  $x/x$  is a maximal element for all  $x \in A$ ,
- 2 the identity  $(x/x)y/y = x/x$  holds in  $A$ , and
- 3 if  $A$  has a top element then the term  $x/x$  is this top element.

## Lemma

Let  $(A, \cdot, /)$  be a narhoop. The following are equivalent.

- 1  $x/x = y/y$  for all  $x, y \in A$ ;
- 2  $(x/x)y = y$  for all  $x, y \in A$ ;
- 3 There exists  $e \in A$  such that  $ey = y$  for all  $y \in A$ .

When these conditions hold, the element  $1 = x/x$  is the maximum left identity element in  $(A, \leq)$ .

A narhoop  $(A, \cdot, /)$  is **unital** if the equivalent conditions of this lemma hold.

In this case we denote by  $1 = x/x$  the distinguished left identity element.

Note that the lemma does not claim that  $1$  is the unique left identity element, even when  $\wedge$  is commutative.

## Theorem

*If  $A$  is finite and unital, then  $1$  is the unique left identity element.*

In a unital narhoop  $A$  the partial order  $\leq$  can be **characterized** in terms of  $1$  and  $/$ :

$$x \leq y \iff x/y = 1$$

for all  $x, y \in A$ .

Furthermore, the left identity  $1$  can also be used to characterize the commutativity of  $\wedge$ .

## Theorem

*Let  $(A, \cdot, /, 1)$  be a unital narhoop. Then:*

- 1 *The left unit  $1$  is the top element of  $(A, \leq)$  if and only if  $\wedge$  is commutative.*
- 2 *The downset  $(1]$  is a subnarhoop and  $((1], \wedge)$  is a semilattice.*

A congruence  $\theta$  on a narhoop  $(A, \cdot, /)$  is said to be **unital** if the factor narhoop  $A/\theta$  is unital.

In other words,  $\theta$  is unital if and only if  $x/x \theta y/y$  for all  $x, y \in A$ .

If  $A$  itself is unital then every congruence on  $A$  is unital.

For a unital congruence on an arbitrary narhoop, set

$$\begin{aligned} N_\theta &= \{x \in A \mid x \theta y/y \text{ for some } y \in A\} \\ &= \{x \in A \mid x \theta y/y \text{ for all } y \in A\}, \end{aligned}$$

where the second equality follows since  $\theta$  is unital.

Analogous to the relationship between congruences and normal subgroups in group theory, we claim that  $\theta$  is determined by the congruence class  $N_\theta$ .

To state the result concisely, we introduce six families of mappings on a narhoop  $(A, \cdot, /)$ .

For each  $x, y \in A$ ,  $i = 1, \dots, 6$ , define  $\phi_{i,x,y} : A \rightarrow A$  by

$$\phi_{1,x,y}(z) = (zx \cdot y)/xy,$$

$$\phi_{2,x,y}(z) = (zx/y)/(x/y),$$

$$\phi_{3,x,y}(z) = (x \cdot zy)/xy,$$

$$\phi_{4,x,y}(z) = (x/zy)/(x/y),$$

$$\phi_{5,x,y}(z) = xy/(x \cdot zy),$$

$$\phi_{6,x,y}(z) = (x/y)/(x/zy).$$

Keeping analogies with group theory in mind, let  $\text{Inn}(A)$  denote the transformation semigroup on  $A$  generated by these six families of mappings.



## Theorem

Let  $\theta$  be a unital congruence on a narhoop  $(A, \cdot, /)$ . Then:

- 1  $N_\theta$  is a subnarhoop of  $A$ ;
- 2 For all  $x, y \in A$ , if  $x \leq y$  and  $x \in N_\theta$ , then  $y \in N_\theta$ ;
- 3  $N_\theta$  is invariant under  $\text{Inn}(A)$ .

Let  $(A, \cdot, /)$  be a narhoop. A nonempty subset  $N$  of  $A$  is said to be a *normal subnarhoop* of  $A$ , denoted  $N \trianglelefteq A$ , if the following hold:

- 1  $N$  is a subnarhoop of  $A$ ;
- 2 For all  $x, y \in A$ , if  $x \leq y$  and  $x \in N$ , then  $y \in N$ ;
- 3  $N$  is invariant under  $\text{Inn}(A)$ .

## Theorem

Let  $(A, \cdot, /)$  be a narhoop and assume  $N \trianglelefteq A$  is nonempty. Define  $\theta_N$  on  $A$  by  $x \theta_N y$  if and only if  $x/y, y/x \in N$ . Then  $\theta_N$  is a unital congruence and  $N_{\theta_N} = N$ .

# Acknowledgments

This research was supported by the automated theorem prover `PROVER9` and the finite model builder `MACE4`, both created by McCune [2009].

We would like to thank Bob Veroff for hosting the 2016 **Workshop on Automated Deduction and its Applications to Mathematics (ADAM)** which is where this collaboration began.

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