Algebras of weakening relations and partially ordered groupoid semantics

Peter Jipsen

School of Computational Sciences and
Center of Excellence in Computation, Algebra and Topology (CECAT)
Chapman University

January 25, 2017
University of Johannesburg Logic Workshop

Outline

- Representable relation algebras
- Abstract relation algebras
- Independence of Tarski's axioms
- Weakening relations
- Involutive residuated lattices
- Heyting relation algebras
- Semantics

Introduction

The **calculus of binary relations** was developed by

Augustus De Morgan [On the syllogism, IV, and on the logic of relations, 1864]

Charles Sanders Peirce [Note B. The logic of relatives, 1883], and

Ernst Schröder [Vorlesungen über die Algebra der Logik, vol. III, Algebra und Logik der Relative, 1895]

At the time it was considered one of the cornerstones of mathematical logic

In 1941 **Alfred Tarski** gave a set of axioms, refined in 1943 to 10 equational axioms, for (abstract) relation algebras

McKinsey [early 1940s] showed the independence of associativity of ";"

Kamel [1950] proved the independence of several other axioms

Representable relation algebras

Let U be a set and $U^2 = U \times U$. The **full set relation algebra on U** is

$\mathrm{Rel}(U) = (\mathcal{P}(U^2), \cup, \cap, \emptyset, U^2, -, ;, Id_U, ^\smile)$

where $R; S = \{(x, y) \in U^2 : \exists z\, (x, z) \in R \text{ and } (z, x) \in S\}$

$-R = U^2 \setminus R$ and

$R^\smile = \{(y, x) : (x, y) \in R\}$ for $R, S \in U^2$

A **small example** of a **full set relation algebra**:

Let $U = \{0, 1\}$. Then $U^2 = \{00, 01, 10, 11\}$ (writing (i, j) as ij)

$\mathrm{Rel}(U)$ has 16 elements, including $\emptyset$, $Id_U = \{00, 11\}$, $-Id_U = \{01, 10\}$

In fact $B = \{\emptyset, Id_U, -Id_U, U^2\}$ is a subalgebra of $\mathrm{Rel}(U)$

Equational properties of representable relation algebras

Rel(U) is a **Boolean algebra with a binary normal operator** ;

a **unary normal operator** $^\smile$ and

a **constant** $Id_U = \{(x,x) : x \in U\}$

This means $R;(S \cup T) = R;S \cup R;T$

$(R \cup S);T = R;T \cup S;T$

$R;\emptyset = \emptyset$ $\qquad\qquad$ $\emptyset;R = \emptyset$

$(R \cup S)^\smile = R^\smile \cup S^\smile$ and $\emptyset^\smile = \emptyset$

The variety RRA

What other properties hold? The class of **representable relation algebras** is defined by RRA $= SP\{Rel(U) : U \text{ is a set}\}$ = all **subalgebras** of **products** of full set relation algebras

Tarski proved RRA is closed under **homomorphic images**, hence a **variety**

For Boolean algebra, Tarski used **Huntington's** [1933] **identities**

(R1) $x \vee y = y \vee x$

(R2) $x \vee (y \vee z) = (x \vee y) \vee z$

(R3) $-(-x \vee y) \vee -(-x \vee -y) = x$

The variety RA of relation algebras

The remaining 7 identities of Tarski's basis are

(R4) $x;(y;z) = (x;y);z$

(R5) $x;1 = x$

(R6) $x^{\smile\smile} = x$

(R7) $(x;y)^{\smile} = y^{\smile};x^{\smile}$

(R8) $(x \vee y);z = x;z \vee y;z$

(R9) $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$

(R10) $x^{\smile};-(x;y) \vee -y = -y$

RA = the variety of algebras that satisfy (R1)-(R10)

Nonrepresentable relation algebras

Tarski [1941] asked: "Is it the case that every sentence of the calculus of relations which is true in every domain of individuals is derivable from the axioms adopted under the second method? This problem presents some difficulties and still remains open. I can only say that I am practically sure that I can prove with the help of the second method all of the hundreds of theorems to be found in Schröder's *Algebra und Logik der Relative*."

D. Monk [1964] showed that RRA is a **nonfinitely axiomatizable subvariety** of RA.

R. McKenzie [1966] found a smallest (16 element) RA that is not in RRA:

$;$	1	a	b	$b^\smile$
1	1	a	b	$b^\smile$
a	a	$-a$	$a \vee b$	$a \vee b^\smile$
b	b	$a \vee b$	b	$\top$
$b^\smile$	$b^\smile$	$a \vee b^\smile$	$\top$	$b^\smile$

where $1, a, b, b^\smile$ are atoms and $\top = 1 \vee a \vee b \vee b^\smile$.
R. Lyndon [1950] and B. Jónsson [1959] found larger examples.

"Short" identities that hold in RRA but not in RA

R. Lyndon [1950] found formulas that hold in RRA but fail in RA, including a shortest one written as identity by R. Maddux (; has precedence over $\wedge$):

$$x \wedge (y \wedge z; t); (u \wedge v; w) \leq z; [(z^{\smile}; x \wedge t; u); w^{\smile} \wedge t; v \wedge z^{\smile}; (x; w^{\smile} \wedge y; v)]; w$$

fails in McKenzie's algebra when $x = z = t = v = w = a, y = b, u = b^{\smile}$

So Tarski's variety RA indeed captures a nice fragment of the nonfinitely axiomatizable theory of binary relations with $\cap, \cup, -, ;, ^{\smile}, id$

Tarski also proved that the **equational theories** of RRA and RA are **undecidable** [1940s]

Independence of (R1)-(R10)

(R1) $x \vee y = y \vee x$

(R2) $x \vee (y \vee z) = (x \vee y) \vee z$

(R3) $-(-x \vee y) \vee -(-x \vee -y) = x$

(R4) $x;(y;z) = (x;y);z$

(R5) $x;1 = x$

(R6) $x^{\smile\smile} = x$

(R7) $(x;y)^{\smile} = y^{\smile};x^{\smile}$

(R8) $(x \vee y);z = x;z \vee y;z$

(R9) $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$

(R10) $x^{\smile};-(x;y) \vee -y = -y$

Joint work with H. Andreka, S. Givant and I. Nemeti [to appear, 2017]

For each (Ri) we show that (Ri) does not follow from the other identities

Need to find an algebra A_i where (Ri) fails and the other identities hold

For (R1) define $A_1 = (\{1, a\}, \vee, -, ;, ^{\smile}, 1)$ where 1 is an identity for ; distinct from a, $a;a = 1$, $x^{\smile} = -x = x$, and $x \vee y = x$

Then (R1) fails since $1 \vee a = 1 \neq a = a \vee 1$

(R2), (R4), (R5), (R6), (R7) are easily seen to hold

(R3) $-(-x \vee y) \vee -(-x \vee -y) = x$ (R9) $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$
(R8) $(x \vee y); z = x; z \vee y; z$ (R10)

Recall $a; a = 1$, $x^{\smile} = -x = x$, and $x \vee y = x$

For (R3): $-(-x \vee y) \vee -(-x \vee -y) = x \vee y \vee x \vee y = x$

For (R8): $(x \vee y); z = x; z = x; z \vee y; z$

For (R9): $(x \vee y)^{\smile} = x^{\smile} = x^{\smile} \vee y^{\smile}$

For (R10): $x^{\smile}; -(x; y) \vee -y = x; (x; y) = y = -y$

In fact any Boolean group with $\vee =$ projection on first argument will work

Independence of Huntington's basis was known, but the argument shows (R2), (R3) **with (R4)-(R10)** do not imply (R1)

Summary of other independence models

$A_2 = \{-1, 0, 1\}$ where $x \vee y = \min(\max(x + y, 1), -1)$ truncated addition

$-$ is subtraction, ; is multiplication, $x^\smile = x$ and $1 = 1$

(R2) fails since $1 \vee (1 \vee -1) = 1 \vee 0 = 1$, but $(1 \vee 1) \vee -1 = 1 \vee -1 = 0$

and it is equally easy to check the other identities hold

$A_3 = \{0, 1\}$ with $\vee =$ join, $-x = x^\smile = x$, $; = \wedge$, and $1 = 1$

Fact 1: For a group G the **complex algebra** $G^+ = (\mathcal{P}(G), \cup, -, ;, ^\smile, \{e\})$ is a (representable) relation algebra where $X; Y = \{xy : x \in X, y \in Y\}$ and $X^\smile = \{x^{-1} : x \in X\}$

For $b \in A$, define the **relativization** $A{\restriction}b = (\{a \wedge b : a \in A\}, \vee, -^b, ;^b, ^\smile, 1)$

where $1 \leq b = b^\smile$, $-^b x = -x \wedge b$ and $x ;^b y = x; y \wedge b$

Fact 2: $A{\restriction}b$ satisfies (R1-3,5-10) and (R4) $\iff$ $b; b \leq b$

$A_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2)^+ \restriction \{(0, 0), (0, 1), (1, 0)\}$ has 8 elements

$A_5 =$ 2-element Boolean algebra with $x; y = 0$, $x^\smile = x$ and $1 = 1$

$A_6 =$ 2-element Boolean algebra with $x; y = x \wedge y$, $x^\smile = 0$ and $1 = 1$

$A_7 = \{\bot, 1, -1, \top\}$ a BA with $x; y = \begin{cases} x & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$, $x^\smile = x$ and $1 = 1$

$A_8 = \{0, 1\}$ a BA with $x; y = \begin{cases} 1 & \text{if } x, y = 0 \\ x \wedge y & \text{otherwise} \end{cases}$, $x^\smile = x$ and $1 = 1$

$A_9 = \mathbb{Z}_3^+$, but for $x \in \{1, 2\}$ and $y, z \in \mathbb{Z}_3$ redefine $\{x\}^\smile = \{x\}$ and $\{x\}; \{y, z\} = \{x^{-1} \cdot y, x^{-1} \cdot z\}$ where $^{-1}$, $\cdot$ are the group operations in $\mathbb{Z}_3$

$A_{10} = \{0, 1\}$ a BA with $; = \vee$, $x^\smile = x$, $1_{RA} = 0$

In each case one needs to check that $A_i \not\models$ (Ri), but the other axioms hold:

(R10) let $x{=}y{=}1$ in $x^\smile; -(x;y) \vee -y = 1 \vee -(1 \vee 1) \vee -1 = 1 \neq 0 = -y$

A variant of Tarski's axioms

Theorem (Andreka, Givant, J., Nemeti)

The identities (R1)-(R10) *are an independent basis for RA.*

Somewhat surprisingly, it turns out that by modifying (R8) slightly, (R7) becomes redundant:

Let $\mathcal{R} =$ (R1)-(R6),(R9),(R10) plus (R8') $= x;(y \vee z) = x;y \vee x;z$

(R1) $x \vee y = y \vee x$

(R2) $x \vee (y \vee z) = (x \vee y) \vee z$

(R3) $-(-x \vee y) \vee -(-x \vee -y) = x$

(R4) $x;(y;z) = (x;y);z$

(R5) $x;1 = x$

(R6) $x^{\smile\smile} = x$

(R7) $(x;y)^{\smile} = y^{\smile};x^{\smile}$

(R8') $x;(y \vee z) = x;y \vee x;z$

(R9) $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$

(R10) $x^{\smile};-(x;y) \vee -y = -y$

Theorem (Andreka, Givant, J., Nemeti)

The identities $\mathcal{R}$ *are also an independent basis for RA.*

Another variant of Tarski's axioms

Let $\mathcal{S}$ = (R1)-(R6),(R8),(R8'),(R10)

(R1) $x \vee y = y \vee x$
(R2) $x \vee (y \vee z) = (x \vee y) \vee z$
(R3) $-(-x \vee y) \vee -(-x \vee -y) = x$
(R4) $x;(y;z) = (x;y);z$
(R5) $x;1 = x$

(R6) $x^{\smile\smile} = x$
(R7) $(x;y)^{\smile} = y^{\smile};x^{\smile}$
(R8) $(x \vee y);z = x;z \vee y;z$
(R8') $x;(y \vee z) = x;y \vee x;z$
(R9) $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$
(R10) $x^{\smile};-(x;y) \vee -y = -y$

Theorem (Andreka, Givant, J., Nemeti)

The identities $\mathcal{S}$ are also an independent basis for RA.

The independence models $A_1 - A_{10}$ are modified somewhat for these proofs.

All models are minimal in size and the paper also describes other models.

Weakening relations

Recall that RA and RRA both have **undecidable** equational theories

I. Nemeti [1987] proved that **removing associativity** from the basis of RA the resulting variety NRA of **nonassociative relation algebras** has a **decidable** equational theory

N. Galatos and J. [2012] showed that if **classical negation** in RA is weakened to a **De Morgan negation** then the resulting variety qRA of **quasi relation algebras** has a **decidable** equational theory

However there are **no natural connections to binary relation** and RRA

Let $\mathbf{P} = (P, \sqsubseteq)$ be a partially ordered set

Let $Q \subseteq P^2$ be an equivalence relation that contains $\sqsubseteq$, and define the set of ***weakening relations*** on $\mathbf{P}$ by $\mathrm{Wk}(\mathbf{P}, Q) = \{\sqsubseteq; R; \sqsubseteq \,:\, R \subseteq Q\}$

Since $\sqsubseteq$ is **transitive and reflexive** $\mathrm{Wk}(\mathbf{P}, Q) = \{R \subseteq Q : \sqsubseteq; R; \sqsubseteq = R\}$

Full weakening relation algebras

If $Q = P \times P$ write $\mathrm{Wk}(\mathbf{P})$ and call it the ***full weakening relation algebra***

Weakening relations are the **analogue of binary relations** when the category **Set** of sets and functions is replaced by the category **Pos** of partially ordered sets and order-preserving functions

Since sets can be considered as **discrete posets** (i.e. ordered by the identity relation), **Pos** contains **Set** as a full subcategory, which implies that weakening relations are a **substantial generalization** of binary relations

However, weaking relations do not allow $-$ or $^\smile$ as operations

They have applications in **sequent calculi**, **proximity lattices/spaces**, **order-enriched categories**, **cartesian bicategories**, **bi-intuitionistic modal logic**, **mathematical morphology** and **program semantics**, e.g. via separation logic

A small example

Let $C_2 = \{0, 1\}$ be the two element chain with $0 \sqsubseteq 1$

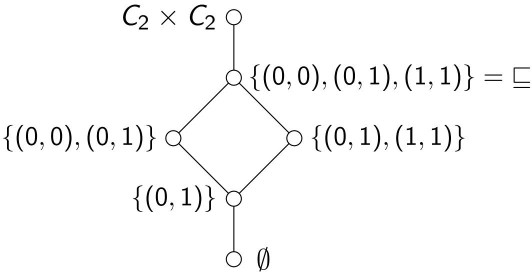

Figure: The full weakening relation algebra $\mathbf{Wk}(\mathbf{C}_2)$

$\mathrm{Wk}(\mathbf{P}, Q)$ is a **complete** and **perfect** distributive lattice under $\cup, \cap$

Operations on weakening relations

Can expand $\mathsf{Wk}(\mathbf{P}, Q)$ to a **Heyting algebra** by adding $\to$

Weakening relations are closed under **composition**: for $R, S \in \mathsf{Wk}(\mathbf{P}, Q)$

$R; S = (\sqsubseteq; R; \sqsubseteq); (\sqsubseteq; S; \sqsubseteq) = \sqsubseteq; (R; \sqsubseteq; S); \sqsubseteq \in \mathsf{Wk}(\mathbf{P}, Q)$

$\sqsubseteq$ is an **identity element** for composition: $R; \sqsubseteq = R = \sqsubseteq; R$

; distributes over arbitrary unions, so we can add **residuals** $\backslash, /$

A ***residuated lattice*** is of the form $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \backslash, /)$ where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and $\backslash, /$ are the **left** and **right residuals** of $\cdot$, i.e., for all $x, y, z \in A$

$$xy \le z \iff y \le x\backslash z \iff x \le z/y$$

So $\mathsf{Wk}(\mathbf{P}, Q)$ is a **distributive residuated lattice** with Heyting implication

Involutive residuated lattices

For an arbitrary constant 0 in a residuated lattice define the ***linear negations*** $\sim x = x\backslash 0$ and $-x = 0/x$

The constant 0 is ***involutive*** if $\sim -x = x = -\sim x$ for all elements x

An ***involutive residuated lattice*** is a residuated lattice with an **involutive** 0

It is ***cyclic*** if $\sim x = -x$

E.g. a **relation algebra** $(A, \wedge, \vee, \neg, ;, ^\smile, 1)$ is a **cyclic involutive residuated lattice** if one defines $x\backslash y = \neg(x^\smile ; \neg y)$, $x/y = \neg(\neg x; y^\smile)$ and $0 = \neg 1$, and omits the operations $\neg, ^\smile$ from the signature

The **cyclic linear negation** is given by $\sim x = \neg(x^\smile) = (\neg x)^\smile$

The variety of **(cyclic) involutive residuated lattices** has a **decidable equational theory** while this is not the case for relation algebras

Heyting relation algebras

A lattice is **bounded** if it has constants $\bot, \top$ such that $\bot \leq x \leq \top$

A **Heyting relation algebra** is a **bounded cyclic involutive lattice with a Heyting arrow** $\to$, defined as the residual of $\wedge$, i.e.,

$$x \wedge y \leq z \iff y \leq x \to z$$

HRA is the **variety** of Heyting relation algebras

Conjecture: HRA has a **decidable** equational theory

Lemma

Let $\mathbf{P} = (P, \sqsubseteq)$ *be a poset,* Q *an equivalence relation that contains* $\sqsubseteq$*,* R *any binary relation on* P *and let* $R' = Q - R$*. Then*

1. $\sqsubseteq \circ R \circ \sqsubseteq = R$ *is equivalent to* $\sqsupseteq \circ R' \circ \sqsupseteq = R'$ *, and*
2. $(\sqsupseteq \circ R \circ \sqsupseteq)'$ *is a weakening relation.*

Proof.

1. Assume $\sqsubseteq \circ R \circ \sqsubseteq = R$ and $(x, y) \in \sqsupseteq \circ R' \circ \sqsupseteq$
Then there exist $(u, v) \in Q$ such that $x \sqsupseteq u$, $(u, v) \notin R$ and $v \sqsupseteq y$
If $(x, y) \in R$ then $u \sqsubseteq x$ and $y \sqsubseteq v$ imply $(u, v) \in R$, which is a contradiction
Hence $(x, y) \in R'$ and therefore $\sqsupseteq \circ R' \circ \sqsupseteq = R'$
The converse is proved by a dual argument □

Proof.

of 2: $(\sqsupseteq \circ R \circ \sqsupseteq)'$ is a weakening relation.
Let $(x, y) \in \sqsubseteq \circ (\sqsupseteq \circ R \circ \sqsupseteq)' \circ \sqsubseteq$
Then there exist $(u, v) \in Q$ such that $x \leq u$, $v \leq y$ and $(u, v) \notin \sqsupseteq \circ R \circ \sqsupseteq$
If $(x, y) \in \sqsupseteq \circ R \circ \sqsupseteq$ then there exist $(r, s) \in R$ such that $r \sqsubseteq x$ and $y \sqsubseteq s$
However, now transitivity implies $r \sqsubseteq u$ and $v \sqsubseteq s$, hence
$(u, v) \in \sqsupseteq \circ R \circ \sqsupseteq$, a contradiction
Therefore $(x, y) \in (\sqsupseteq \circ R \circ \sqsupseteq)'$, so $\sqsubseteq \circ (\sqsupseteq \circ R \circ \sqsupseteq)' \circ \sqsubseteq \subseteq (\sqsupseteq \circ R \circ \sqsupseteq)'$
The reverse inclusion always holds by reflexivity. □

Theorem

$\mathbf{Wk}(\mathbf{P}, Q) = (\mathrm{Wk}(\mathbf{P}, Q), \cap, \cup, \to, Q, \emptyset, \circ, \sqsubseteq, \backslash, /, 0)$ *is a Heyting relation algebra.* *In particular,* $\top = Q$, $\bot = \emptyset$,
$R \to S = (\sqsupseteq \circ (R \cap S') \circ \sqsupseteq)'$,
$R\backslash S = (\sqsupseteq \circ R^{\smile} \circ S' \circ \sqsupseteq)'$,
$R/S = (\sqsupseteq \circ R' \circ S^{\smile} \circ \sqsupseteq)'$,
the involutive constant is $0 = \sqsupseteq'$, *and* $\sim R = R^{\smile\prime} = R'^{\smile}$.

Proof. $\mathbf{Wk}(\mathbf{P}, Q)$ has the empty set and is closed under ; meets, joins
$'$ is complementation, but it is not an operation on $\mathrm{Wk}(\mathbf{P}, Q)$
The relation $\sqsubseteq$ is the unit for weakening relations since $\sqsubseteq \circ \sqsubseteq = \sqsubseteq$
The formula for $R \to S$ is justified by the preceeding lemma and:

$$\begin{aligned} R \cap S \subseteq T &\iff R \cap T' \cap S = \emptyset \iff R \cap T' \subseteq S' \\ &\iff R \cap T' \subseteq (\sqsupseteq \circ (R \cap T') \circ \sqsupseteq) \subseteq (\sqsupseteq \circ S' \circ \sqsupseteq) = S' \\ &\iff S \subseteq (\sqsupseteq \circ (R \cap T') \circ \sqsupseteq)' \iff S \subseteq R \to T. \end{aligned}$$

For $R\backslash S, R/S, \sim R$ the calculations are similar □

If $\mathbf{P}$ is a discrete poset then $\mathbf{Wk}(\mathbf{P})$ is the full representable relation algebra on the set P

$\implies$ algebras of weakening relations are similar to representable relation algebras

Define the class RHRA of ***representable Heyting relation algebras*** as all algebras that are **embedded** in a weakening algebra $\mathbf{Wk}(\mathbf{P}, Q)$ for some poset $\mathbf{P}$ and equivalence relation Q that contains $\sqsubseteq$

In fact the variety RRA is a finitely axiomatizable subclass of RHRA

Theorem

1. RHRA *is closed under subalgebras and products.*
2. RRA *is the subclass of algebras in* RHRA *that satisfy* $\neg\neg x = x$, *i.e., have Boolean algebra reducts.*
3. RHRA *is not finitely axiomatizable relative to the variety* HRA

Currently it is not known if RHRA is closed under homomorphic images

Poset semantics of weakening relations

Birkhoff showed that a finite distributive lattice $\mathbf{A}$ is determined by its poset $J(\mathbf{A})$ of completely join-irreducible elements (with the order induced by $\mathbf{A}$)

The result also holds for complete perfect distributive lattices

Conversely, if $\mathbf{Q} = (Q, \leq)$ is a poset, then the set of downward closed subsets $D(\mathbf{Q})$ of $\mathbf{Q}$ forms a complete perfect distributive lattice under intersection and union

$D(\mathbf{Q})$ is a Heyting algebra, with $U \to V = Q - {\uparrow}(U - V)$ for any $U, V \in D(\mathbf{Q})$

For a poset $\mathbf{P}$, Wk($\mathbf{P}$) is complete and perfect and $J(\mathrm{Wk}(\mathbf{P})) \cong \mathbf{P} \times \mathbf{P}^\partial$

The composition ; of $\mathbf{Wk}(\mathbf{P})$ is determined by its restriction to pairs of $\mathbf{P} \times \mathbf{P}^\partial$, where it is a partial operation given by

$$(t, u) \circ (v, w) = \begin{cases} (t, w) & \text{if } u = v \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Semantics of relation algebras

For comparison, we first consider the case of relation algebras.

A complete perfect relation algebra has a complete atomic Boolean algebra as reduct, and the set of join-irreducibles is the set of atoms.

B. Jónsson and A. Tarski [1952] showed the operation of composition, restricted to atoms, is a partial operation precisely when the atoms form a **(Brandt) groupoid**, or equivalently a **small category with all morphism being invertible**.

For Heyting relation algebras we have a similar result using **partially-ordered groupoids**

Groupoids and partially ordered groupoids

A ***groupoid*** is defined as a **partial** algebra $\mathbf{G} = (G, \circ, ^{-1})$ such that $\circ$ is a partial binary operation and $^{-1}$ is a (total) unary operation on G that satisfy:

1. $x \circ y, y \circ z \in G \implies (x \circ y) \circ z = x \circ (y \circ z)$,
2. $x \circ y \in G \iff x^{-1} \circ x = y \circ y^{-1}$,
3. $x \circ x^{-1} \circ x = x$ and $x^{-1-1} = x$.

These axioms imply $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$

Typical examples of groupoids are **disjoint unions of groups** and the ***pair-groupoid*** $(X \times X, \circ, ^{\smile})$

Partially ordered groupoids and Heyting relation algebras

A ***partially-ordered groupoid*** $(G, \leq, \circ, ^{-1})$ is a groupoid $(G, \circ, ^{-1})$ such that $(G, \leq)$ is a poset and $\circ, ^{-1}$ are order-preserving:

- $x \leq y$ and $x \circ z, y \circ z \in G \implies x \circ z \leq y \circ z$
- $x \leq y$ and $z \circ x, z \circ y \in G \implies z \circ x \leq z \circ y$
- $x \leq y \implies y^{-1} \leq x^{-1}$

If $\mathbf{P} = (P, \leq)$ a poset then $\mathbf{P} \times \mathbf{P}^\partial = (P \times P, \leq, \circ, ^\smile)$ is a partially-ordered groupoid with $(a, b) \leq (c, d) \iff a \leq c$ and $d \leq b$.

Theorem

Let $\mathbf{G} = (G, \leq, \circ, ^{-1})$ be a partially-ordered groupoid. Then $D(\mathbf{G})$ is a Heyting relation algebra.

Semantics for full weakening relation algebras

In fact for a poset $\mathbf{P} = (P, \sqsubseteq)$ the weakening relation algebra $\mathbf{Wk}(\mathbf{P})$ is obtained from the partially-ordered pair-groupoid $\mathbf{G} = \mathbf{P} \times \mathbf{P}^\partial$

For example, the 3-element chain $\mathbf{C}_3$ gives a 9-element partially ordered groupoid, and $\mathbf{Wk}(\mathbf{C}_3)$ has 20 elements

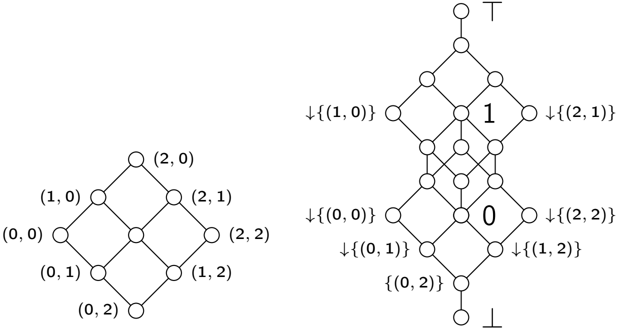

Figure: The weakening relation $\mathbf{Wk}(\mathbf{C}_3)$ and its po-pair-groupoid

Cardinality of $\mathbf{Wk}(\mathbf{C}_n)$

Theorem

For an n-element chain $\mathbf{C}_n$ *the weakening relation algebra* $\mathbf{Wk}(\mathbf{C}_n)$ *has cardinality* $\binom{2n}{n}$.

Proof.

This follows from the observation that $D(\mathbf{C}_m \times \mathbf{C}_n)$ has cardinality $\binom{m+n}{n}$
For $n = 1$ this holds since an m-element chain has $m + 1$ down-closed sets
Assuming the result holds for n, note that $\mathbf{P} = \mathbf{C}_m \times \mathbf{C}_{n+1}$ is the disjoint union of $\mathbf{C}_m \times \mathbf{C}_n$ and $\mathbf{C}_m$, where we assume the additional m elements are not below any of the elements of $\mathbf{C}_m \times \mathbf{C}_n$
The number of downsets of $\mathbf{P}$ that contain an element a from the extra chain $\mathbf{C}_m$ as a maximal element is given by $\binom{k+n}{n}$ where k is the number of elements above a
Hence the total number of downsets of $\mathbf{P}$ is $\sum_{k=0}^{m} \binom{k+n}{n} = \binom{m+n+1}{n+1}$. □

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Thank You