Algebraic cut elimination for Residuated Lattices

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Lattices with operators

Gehrke and Harding [2001] develop canonical extensions for **lattices with operators**

Dunn, Gehrke, Palmigiano [2005] define **generalized Kripke frames** using (maximally disjoint) **filter–ideal pairs**

For the lattice reducts, this is based on G. Birkhoff's **polarities**, A. Urquhart's **lattice spaces** and the notion of **contexts** from R. Wille's **Formal Concept Analysis**

Expansions of residuated lattices **by operators** fit into this theory

However, integrating the **proof theory** of residuated lattices and their **reducts/expansions** requires further ideas

A glimpse of algebraic proof theory

Gentzen [1936] defined **sequent calculi**, including **LK** (for classical logic) and **LJ** (for intuitionistic logic)

For **proof search** and **proof normalization**, he proved that the **cut rule** can be **omitted** without affecting provability

Example: A a **sequent calculus** for residuated lattices

Let **RL** be the **equational theory** of **residuated lattices**

Let $T = Fm_{\vee,\wedge,\cdot,1,\backslash,/}(x_1, x_2, \ldots)$, $\quad W = F_{Mon(\circ,\varepsilon)}(T)$, $\quad W' = U \times T$

where $U = \{u \in F_{Mon(\circ,\varepsilon)}(T \cup \{x_0\}) : u$ contains **exactly one** $x_0\}$

The Gentzen system GL

A **Horn formula** $\varphi_1 \& \cdots \& \varphi_n \to \psi$ is written $\frac{\varphi_1 \ \cdots \ \varphi_n}{\psi}$

Let $a, b, c \in T$, $s, t \in W$ and $u \in U$

GL:

$$\frac{}{a \Rightarrow a} \qquad \frac{t \Rightarrow a}{t \Rightarrow a \vee b} \qquad \frac{t \Rightarrow b}{t \Rightarrow a \vee b} \qquad \frac{u(a) \Rightarrow c \quad u(b) \Rightarrow c}{u(a \vee b) \Rightarrow c}$$

$$\frac{t \Rightarrow a \quad u(a) \Rightarrow b}{u(t) \Rightarrow b}(\text{cut}) \qquad \frac{u(a) \Rightarrow c}{u(a \wedge b) \Rightarrow c} \qquad \frac{u(b) \Rightarrow c}{u(a \wedge b) \Rightarrow c} \qquad \frac{t \Rightarrow a \quad t \Rightarrow b}{t \Rightarrow a \wedge b}$$

$$\frac{u(a \circ b) \Rightarrow c}{u(a \cdot b) \Rightarrow c} \qquad \frac{s \Rightarrow a \quad t \Rightarrow b}{s \circ t \Rightarrow a \cdot b} \qquad \frac{}{\varepsilon \Rightarrow 1} \qquad \frac{u(\varepsilon) \Rightarrow a}{u(1) \Rightarrow a}$$

$$\frac{a \cdot t \Rightarrow b}{t \Rightarrow a \backslash b} \qquad \frac{t \Rightarrow a \quad u(b) \Rightarrow c}{u(t \circ (a \backslash b)) \Rightarrow c} \qquad \frac{t \cdot b \Rightarrow a}{t \Rightarrow a / b} \qquad \frac{t \Rightarrow b \quad u(a) \Rightarrow c}{u((a/b) \circ t) \Rightarrow c}$$

Example of a *cut-free* **RL** proof

$$\dfrac{\dfrac{\dfrac{\dfrac{z \Rightarrow z \quad x \Rightarrow x}{z \circ (z \backslash x) \Rightarrow x}}{z \circ (z \backslash x \wedge z \backslash y) \Rightarrow x} \quad \dfrac{\dfrac{z \Rightarrow z \quad y \Rightarrow y}{z \circ (z \backslash y) \Rightarrow y}}{z \circ (z \backslash x \wedge z \backslash y) \Rightarrow y}}{z \circ (z \backslash x \wedge z \backslash y) \Rightarrow x \wedge y}}{z \backslash x \wedge z \backslash y \Rightarrow z \backslash (x \wedge y)}$$

Semantics of sequent calculi: Residuated frames

Let $\mathbf{GL}_{cf}$ be the sequent calculus **GL without** the **cut rule**

Define a binary relation $N \subseteq W \times W'$ by

$$wN(u, a) \iff u(w) \Rightarrow a \quad \text{is provable in } \mathbf{GL}_{cf}$$

Define the **accessibility** relations $R_\circ \subseteq W^3$, $R_{\backslash\backslash}, R_{//}$ by

$$R_\circ(v_1, v_2, w) \iff v_1 \circ v_2 = w$$

$$R_{\backslash\backslash} = \{((u, a), x, (u(\_\circ x), a)) : u \in U, a \in T, x \in W\}$$

$$R_{//} = \{(x, (u, a), (u(x \circ \_), a)) : u \in U, a \in T, x \in W\}$$

Then $(W, W', N, R_\circ, R_{\backslash\backslash}, R_{//})$ is a **residuated frame**

(A **general** residuated frame is $(W, W', N, R_i(i \in I))$)

Algebraic cut-admissibility

Theorem

[Okada, Terui 1999, Galatos, J. 2013]. The following are equivalent:

1. $t \Rightarrow a$ *is provable in* $\boldsymbol{GL}$
2. $t \le a$ *holds in* $\boldsymbol{RL}$
3. $t \Rightarrow a$ *is provable in* $\boldsymbol{GL}_{cf}$

Proof (outline): (3⇒1) is obvious. (1⇒2) Assume $t \Rightarrow a$ is provable **with cut**. Show that **all sequent rules** hold as quasiequations in **RL** (where $\Rightarrow, \circ$ are **replaced by** $\le, \cdot$)

(2⇒3) Assume $t \le a$ holds in **RL** and define an algebra $\mathbf{W}^+ = (C[\mathcal{P}(W)], \cup, \cap, \cdot, 1, \backslash, /)$ using the **closed sets** $C(X)$ of the **polarity** (W, W', N) and

$$X \cdot Y = C(\{w : R(v_1, v_2, w) \text{ for some } v_1 \in X, v_2 \in Y\})$$

$$X\backslash Y = \{w \in W : X \cdot \{w\} \subseteq Y\} \qquad Y/X = \{w \in W : \{w\} \cdot X \subseteq Y\}.$$

Proof outline (continued)

Then $\mathbf{W}^+$ is a residuated lattice, hence satisfies $t \leq a$

Let $f : T \to \mathbf{W}^+$ be a **homomorphism**

Extend to $\bar{f} : W \to \mathbf{W}^+$, so $t \leq a$ implies $\bar{f}(t) \subseteq \bar{f}(a)$

Define $\{b\}^{\triangleleft} = \{w \in W : wN(x_0, b)\}$

Prove by **induction** that $b \in \bar{f}(b) \subseteq \{b\}^{\triangleleft}$ for all $b \in T$

Then $t \in \bar{f}(t) \subseteq \bar{f}(a) \subseteq \{a\}^{\triangleleft}$, hence $tN(x_0, a)$

Therefore $\quad t \Rightarrow a \quad$ holds in $\mathbf{GL}_{cf}$ □

Theorem

The equational theory of residuated lattices is ***decidable****. Moreover,* **RL** *has the* ***finite model property***
[Galatos, J. 2013] The variety of integral RL (i.e., $x \wedge 1 = x$) has the ***finite embedding property****, hence the* ***universal theory is decidable****.*

Expanding this approach to GBI-algebras

A similar approach can be used to prove that the equational theory of GBI-algebras is decidable

Add Gentzen rules for an external connective $\owedge$ corresponding to $\wedge$, and rules for $\to$

Expand the residuated frame with a ternary relation for $\owedge$

Theorem

*[Galatos, J.] The equational theory of GBI-algebras is **decidable**. Moreover, (G)BI-algebras have the **finite model property***

Theorem

*[Galatos, J.] The variety of integral GBI-algebras (i.e., $x \wedge 1 = x$) has the **finite embedding property**, hence the **universal theory is decidable**.*

Conclusion

Substructural logics and **residuated lattices** are an excellent **framework** for investigating and **comparing** propositional logics

By considering **expansions** many more propositional logics are covered

Algebraic, **semantic** and **proof theoretic** techniques can often be adapted to the **expansions**

Some References

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Thank You