Effect algebras, involutive residuated posets, and a brief introduction to the LEAN theorem prover

Peter Jipsen
joint work with Sarah Alexander and Nadiya Upegui

Chapman University

University of Johannesburg Logic Workshop
January 30, 2018

Outline

Logic

Logic provides frameworks for **precise reasoning**

Studies **implications** ($\implies$) between **propositions**

Ususally $P \implies P$, and

if $P \implies Q$ and $Q \implies R$ then $P \implies R$

$P \implies Q$ and $Q \implies P$ if and only if $P \iff Q$

Then $\implies$ is a **partial order** between equivalence classes

Syntax: $\implies$, and, or, negation, quantifiers $(\forall, \exists), \ldots$ for formulas

Semantics (meaning) is given by mathematical structures

Algebra

Algebra is about **properties of operations** $f : A^n \to A$ and

solving (systems of) **equations**, like $a \cdot x = b$

Uniform replacement of expressions leads to **equational logic**

Example: $(\mathbb{N}, +, 0)$ is a **cancellative commutative monoid**

(comm) $x + y = y + x$,

(canc) $x + y = x + z \implies y = z$,

(asso) $(x + y) + z = x + (y + z)$ and (iden) $x + 0 = x$

Naturally ordered by $\quad x \le y \quad \iff \quad \exists w \; x + w = y$

$(\mathbb{N}, \le)$ is a linearly ordered set (in fact well ordered)

Algebraic logic

This is about the interplay between **algebra and logic**, based on a correspondence between **partially ordered algebras** and **logical theories**

Partial order $\leq$ corresponds to $\implies$

Greatest lower bound $\wedge$ (meet) corresponds to **and**

Least upper bound $\vee$ (join) corresponds to **or**

Subvarieties of algebras correspond to **extensions of logical theories**

Groups versus residuated posets

$(G, \cdot, \backslash, /)$ is a **group** if $\cdot$ is associative binary operation

and for all $x, y, z \in G$, $\quad x \cdot y = z \iff x = z/y \iff y = x\backslash z$

i.e., $x \cdot y = z$ has **unique solutions** for x and y

The identity is $e = x/x$ and $x^{-1} = e/x$

But the natural order $x \leq y \iff \exists w\ x \cdot w = y$ relates all elements

$(A, \leq, \cdot, \backslash, /)$ is a **residuated poset** if $\leq$ is a partial order, $\cdot$ is associative

and for all $x, y, z \in A$, $\quad x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z$

i.e., $x \cdot y \leq z$ has (unique) **greatest solutions** for x and y

So residuated posets are the **analogue** of groups

Cancellative monoids with zero

Cancellativity is too strong for finite monoids (and semigroups):

Lemma

*Any finite **cancellative** monoid is a **group***

$(M, \cdot, e, 0)$ is a **monoid with zero** if $x \cdot 0 = 0 = 0 \cdot x$

For monoids with a zero element define (weak) **cancellativity** by

$x \neq 0$ and $x \cdot y = x \cdot z \implies y = z$, and

$z \neq 0$ and $x \cdot z = y \cdot z \implies x = y$

Again define $x \leq y \iff \exists w \; x \cdot w = y$

A monoid is **positive** if $x \cdot y = e \implies x = e = y$.

Cancellative positive monoids with zero

Lemma

Let $(M, \cdot, e, 0)$ be a cancellative positive monoid with zero.
Then $(M, \leq)$ is a poset, and $y \leq z \implies xy \leq xz$.

Proof.

Reflexivity $(x \leq x)$ holds since $xe = x$.
Assume $x \leq y$ and $y \leq z$. Then $xu = y$ and $yv = z$ for some u, v.
Hence $(xu)v = x(uv) = z$, so $x \leq z$.
Assume $x \leq y$ and $y \leq x$. Then $xu = y$ and $yv = x$ for some u, v.
Now $xuv = x = xe$, so $uv = e$ by cancellativity.
Therefore $u = e$ by positivity, hence $x = xe = y$.
Finally, assume $y \leq z$. Then $yw = z$ for some w.
Therefore $xyw = xz$ and hence $xy \leq xz$. □

Conjugation

What if the natural order is defined by $x \le y \iff \exists w\ wx = y$?

For commutative monoids this **makes no difference**

In the **noncommutative** case the following axiom gives the same poset:

Conjugation: $\exists w\ xw = y \iff \exists w\ wx = y$

Now it also follows that $\cdot$ is order-preserving in both arguments

A **generalized pseudo-effect algebra** is a cancellative conjugated positive monoid with zero

These algebras are usually defined on $A \setminus \{0\}$ as a **partial algebra**

Then xy is **undefined** iff $xy = 0$ in A

What is a Partial Algebra?

- A *partial operation* g of arity n on a set A is a function from a subset $D(g)$ of A^n to A.
- The notation $g : A^n \multimap A$ is used to indicate that g is an n-ary partial function on A.
- A *partial algebra* is a pair $\mathbf{A} = (A, \mathcal{F}^{\mathbf{A}})$ where A is a set and $\mathcal{F}^{\mathbf{A}}$ is a set of operations on A containing at least one partial operation.

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	-
2	2	3	-	-
3	3	-	-	-

Generalized Pseudo-Effect Algebras

A **generalized pseudo-effect algebra** (or GPE-algebra) $\mathbf{A} = (A, +, 0)$ is a partial algebra such that for all $x, y, z \in A$

(asso) If $x + y$ and $(x + y) + z$ are defined then $y + z$ is defined and $(x + y) + z = x + (y + z)$

(iden) $x + 0 = x = 0 + x$

(canc) If $x + y$ is defined,
$x + y = x + z \implies y = z$ and $x + y = z + y \implies x = z$

(conj) $\exists z(x + z = y) \iff \exists w(w + x = y)$

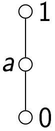
(posi) $x + y = 0 \implies x = 0 = y$

GPE-algebras are **partially ordered** by

$$x \leq y \iff \exists w \; x + w = y$$

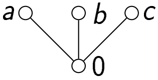
Examples of GPE-Algebras

GPEA of size 3



$+$	0	a	1
0	0	a	1
a	a	1	$-$
1	1	$-$	$-$

GPEA of size 4



$+$	0	a	b	c
0	0	a	b	c
a	a	$-$	$-$	$-$
b	b	$-$	$-$	$-$
c	c	$-$	$-$	$-$

Subalgebras of GPEA's are not always GPEA's

$\mathbf{A} = (A, +, 0)$

$+$	0	a	b	c	1
0	0	a	b	c	1
a	a	$-$	1	$-$	$-$
b	b	$-$	$-$	1	$-$
c	c	1	$-$	$-$	$-$
1	1	$-$	$-$	$-$	$-$

This is a GPE-aglebra.

$\mathbf{B} = (A \setminus \{c\}, +\restriction_B, 0)$

$+\restriction_B$	0	a	b	1
0	0	a	b	1
a	a	$-$	1	$-$
b	b	$-$	$-$	$-$
1	1	$-$	$-$	$-$

This is a closed subalgebra of **A** that now fails conjugation.

Downward Closed Subsets of GPE-Algebras

Lemma

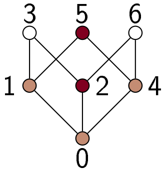
Let $\mathbf{A} = (A, +, 0)$ *be a GPE-algebra and* $0 \in B \subseteq A$.

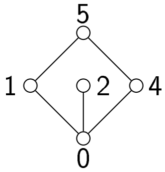
Define the downward closed subset $\downarrow B$ *of* B *by*

$\downarrow B := \{x \in A \mid x \leq y \text{ for some } y \in B\}$.

Then $\mathbf{B} = (\downarrow B, +\restriction_{\downarrow B}, 0)$ *is a GPE-algebra.*

Example of a Downward Closed Subset





+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	—	3	—	5	—	—
2	2	3	—	—	6	—	—
3	3	—	—	—	—	—	—
4	4	5	6	—	—	—	—
5	5	—	—	—	—	—	—
6	6	—	—	—	—	—	—

+	0	1	2	4	5
0	0	1	2	4	5
1	1	—	—	5	—
2	2	—	—	—	—
4	4	5	—	—	—
5	5		—	—	—

Proof of Downward Closed Subsets

For simplicity of notation, we will denote $+\restriction_B$, the restriction of $+$ onto B, as $+_B$.

(i) **Identity** 0
$\forall x \in {\downarrow}B, x + 0 = x$ [identity]
$0 \leq x$ [defn. partial order]
$0 \in {\downarrow}B$ [defn. of ${\downarrow}B$]
$0 +_B x = x +_B 0 = x$, trivially.

(ii) **Cancellativity**
Assume $x +_B y = x +_B z$ for some $x, y, z \in {\downarrow}B$.
$x + y = x + z$
$x = y$ [canc. in **A**]
The same can be shown for right cancellativity.

The last three axioms of positivity, associativity, and conjugation can be proven as well.

Products of GPE-algebras

Definition

The *direct product* of a family $\{\mathbf{A}_i \mid i \in I\}$ of GPE-algebras with $\mathbf{A}_i = (A_i, +_i, 0_i)$ is the set of *choice functions*

$$\prod_{i\in I} \mathbf{A}_i = \{f : I \to \bigcup_{i\in I} A_i \mid f(i) \in A_i \text{for all} i \in I\}$$

We further define $\prod_{i\in I} \mathbf{A}_i = (\prod_{i\in I} A_i, +, 0)$ such that:

- $\mathbf{0}(i) := 0_i$
- $(\mathbf{a} + \mathbf{b})(i) := a_i +_i b_i$

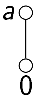
Products of GPE-aglebras (continued)

Lemma

The direct product of a family of GPE-algebras is also a GPE-algebra.

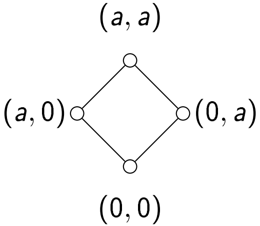
A

$+$	0	a
0	0	a
a	a	$-$



A $\times$ **A**

$+$	$(0,0)$	$(0,a)$	$(a,0)$	(a,a)
$(0,0)$	$(0,0)$	$(0,a)$	$(a,0)$	(a,a)
$(0,a)$	$(0,a)$	$-$	(a,a)	$-$
$(a,0)$	$(a,0)$	(a,a)	$-$	$-$
(a,a)	(a,a)	$-$	$-$	$-$



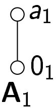
Simple Pastings of GPE-algebras (continued)

Lemma

The simple pasting of a family of GPE-algebras is also a GPE-algebra.

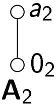
$\mathbf{A}_1$

$+_1$	0_1	a_1
0_1	0_1	a_1
a_1	a_1	—



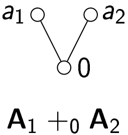
$\mathbf{A}_2$

$+_2$	0_2	a_2
0_2	0_2	a_2
a_2	a_2	—



$\mathbf{A}_1 +_0 \mathbf{A}_2$

$+$	0	a_1	a_2
0	0	a_1	a_2
a_1	a_1	—	—
a_2	a_2	—	—



Why Study Effect Algebras?

Effect algebras have applications in the foundations of **quantum mechanics** and in **probability theory**.

D. J. Foulis and M. K. Bennett [1994]:

If a quantum-mechanical system $\mathcal{S}$ is represented in the usual way by a Hilbert space $\mathcal{H}$, then a self-adjoint operator A on $\mathcal{H}$ such that $\mathbf{0} \leq \mathbf{A} \leq \mathbf{1}$ corresponds to an **effect** for $\mathcal{S}$. Effects are of significance in representing **unsharp** measurements or observations on the system $\mathcal{S}$, and effect valued measures play an important role in stochastic quantum mechanics.

Effect Algebras

An **effect algebra** $\mathbf{A} = (A, +, ', 0, 1)$ is a partial algebra with a partial binary operation $+$ and a total unary operation $'$ such that for all $x, y, z \in A$

- (com) if $x + y$ is defined then $x + y = y + x$ (**commutative**)
- (asso) if $x + y$ and $(x + y) + z$ are defined then $y + z$ is defined and $(x + y) + z = x + (y + z)$ (**associative**)
- (orth) $x + y = 1$ if and only if $y = x'$ (**com-orthocomplement**)
- (posi) $x + y = 0$ implies $x = 0$ (**positive**)

Consequences of the Effect Algebra Axioms

Lemma

For all $x, y, z \in A$, effect algebras also satisfy

(dbl) $x = x''$ (**double orthocomplement**)

(canc) If $x + y$ is defined,
$x + y = x + z \implies y = z$ (**cancellative**)

(iden) $x + 0 = x = 0 + x$ (**identity**)

(0-1) if $x + 1$ is defined then $x = 0$ (**zero-one**)

Hence they are **naturally ordered**

Consequences of the Effect Algebra Axioms

Proof

$x + x' = 1$ [orth]
$x' + x = 1$ [com]
$x = x''$ [orth]
$\therefore$ **Double Orthocomplement**
Assume $x + z = y + z$ is defined.
$\exists! w$ s.t. $(x + z) + w = 1 = (y + z) + w$ [orth]
$x + (z + w) = 1 = y + (z + w)$ [assoc]
$(z + w) = x' = y'$ [orth]
$x = x'' = y'' = y$ [dbl]
$\therefore$ **Cancellative**

Examples of an Effect Algebras

One example of an effect algebra is the **standard effect algebra** $[0,1]_E = ([0,1], +, ', 0, 1)$ where

$$x + y = \begin{cases} x + y \text{ under standard addition} & \text{if } x + y \leq 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Generalized Pseudo-Effect Algebras

A **generalized pseudo-effect algebra** $\mathbf{A} = (A, +, 0)$ is a partial algebra such that for all $x, y, z \in A$

- (asso) If $x + y$ and $(x + y) + z$ are defined then $y + z$ is defined and $(x + y) + z = x + (y + z)$
- (iden) $x + 0 = x = 0 + x$
- (canc) If $x + y$ is defined, $x + y = x + z \implies y = z$ and $x + y = z + y \implies x = z$
- (conj) $\exists z(x + z = y) \iff \exists w(w + x = y)$
- (posi) $x + y = 0 \implies x = 0 = y$

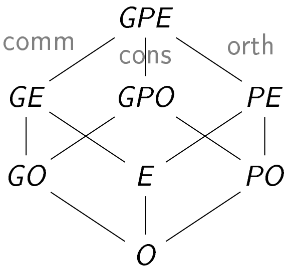
These algebras are not necessarily commutative, have no orthocomplement, and no constant 1.

Can still define a poset by $x \leq y \iff \exists z(x + z = y)$

Subclasses and Expansions of GPE-algebras

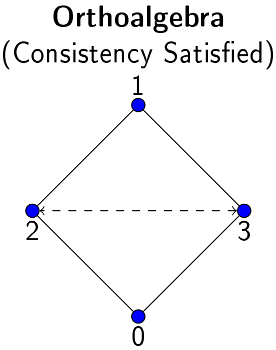
Adding combinations of three independent axioms creates subclasses:

(com) $x + y = y + x$ (**commutative**)
(orth) $x + y = 1 \iff y = x^{\sim} \iff x = y^{-}$ (**orthocomplement**)
(cons) $x + x$ defined $\implies x = 0$ (**consistent**)

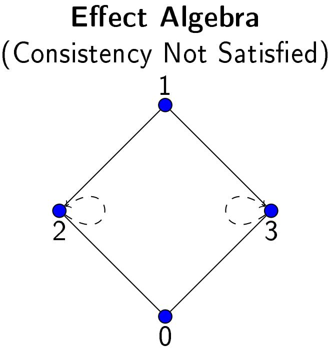


G = Generalized, P = Pseudo, E = Effect Algebras, O = Orthoalgebras

Examples



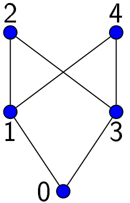
Orthoalgebra
(Consistency Satisfied)
1
2
3
0



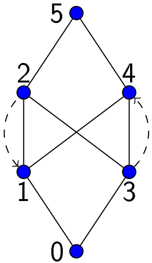
Effect Algebra
(Consistency Not Satisfied)
1
2
3
0

Examples

Generalized Effect Algebra
(Orthocomplementation Not Satisfied)

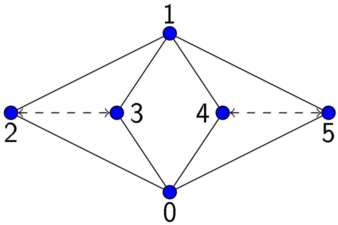


Effect Algebra
(Orthocomplementation Satisfied)

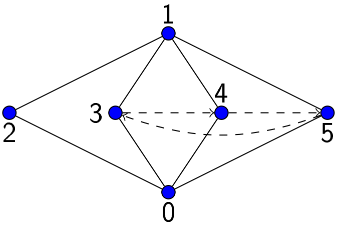


Examples

Orthoalgebra
(Commutativity Satisfied)



Pseudo Orthoalgebra
(Commutativity Not Satisfied)



Residuated posets from GPE-algebras

Definition

Let $\mathbf{A} = (A, +, 0)$ be a generalized pseudo-effect algebra.
Define $\overline{\mathbf{A}} = (A \cup \{\bot, \top\}, \cdot, e, \backslash, /)$ as follows:

- $e = 0$ and $\bot < x < \top$ for any $x \in A$
- $x \cdot y := x + y$ if $x + y$ is defined, else $x \cdot y = \top$ for $x, y \in A$
- $y/x = z \iff y = z + x$ and $x\backslash y = z \iff y = x + z$
- $y/x = x\backslash y = \bot$ if $x \not\leq y$
- $\bot/x = x\backslash\bot = x/\top = \top\backslash x = \bot$
- $\bot x = x\bot = \bot$ and $y\top = \top y = \top$ for $x, y \in A \cup \{\bot, \top\}$ with $y \neq \bot$

Theorem

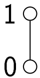
(Rump-Yang 2014) Let $\mathbf{A}$ *be a GPE-algebra. Then* $\overline{\mathbf{A}}$ *is a residuated poset.*

Corollary

Every GPE-algebra is an interval in some (total!) residuated poset.
A GPE-algebra $\mathbf{A}$ *is lattice-ordered iff* $\bar{\mathbf{A}}$ *is a residuated lattice.*

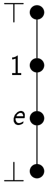
E_2

$\oplus$	0	1
0	0	1
1	1	–



$\bar{E}_2$

$\cdot$	$\bot$	0	1	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
0	$\bot$	0	1	$\top$
1	$\bot$	1	$\top$	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$



Pseudo-effect algebras and involutive residuated lattices

It is also possible to axiomatize the residuated po-monoids that uniquely correspond to GPE-algebras

Pseudo-effect algebras and involutive residuated lattices

It is also possible to axiomatize the residuated po-monoids that uniquely correspond to GPE-algebras

A residuated poset is **involutive** if there exists an element d such that the terms ${\sim}x = x\backslash d$ and $-x = d/x$ satisfy $-{\sim}x = x = {\sim}{-}x$. Then $-d = e = {\sim}d$ and $x\backslash y = {\sim}((-y)x)$, $x/y = -(y({\sim}x))$.

Equivalently, $(A, \leq, \cdot, e, d, \sim, -)$ is an involutive residuated po-monoid if $-{\sim}x = x = {\sim}{-}x$ and $xy \leq d \iff x \leq -y$.

Theorem

If $\mathbf{A}$ *is a PE/PO-algebra, effect algebra or orthoalgebra, then* $\overline{\mathbf{A}}$ *is an involutive residuated poset.*

Building GPE-algebras

Theorem

Let $\mathbf{P} = (P, \oplus, 0)$ *be a GPEA. Let* $P_m = P \cup \{m\}$ *where* $m \notin P$. *Then* $\mathbf{P}_m = (P_m, +, 0)$ *is a GPEA if and only if the following conditions hold:*

(1) *For all* $x, y \in P$, $x + y \in P$ *iff* $x \oplus y$ *is defined, in which case* $x + y = x \oplus y$.

(2) $m + 0 = m = 0 + m$

(3) $m + x$ *and* $x + m$ *are undefined for all* $x \in P_m \setminus \{0\}$

(4) $x + y = m = x + z \implies y = z$ *and* $x + y = m = z + y \implies x = z$

(5) *For all* $x, y \in P$, $x + y = m \implies \exists u, v$ *s.t.* $u + x = m = y + v$

(6) $(x + y) + z = m \iff x + (y + z) = m$

Enumerating GPE-algebras: Initial Setup

Consider a GPE-algebra $\mathbf{P} = (P, \oplus, 0)$.

The program generates a new GPE-algebra $\mathbf{P_m} = (P_m, +, 0)$, with $P_m = P \cup \{m\}$.

Initial rules for $+$:

(1) For all $x, y \in P$, $x + y$ is defined iff $x \oplus y$ is defined, in which case $x + y = x \oplus y$ **(satisfies posi)**

(2) $m + 0 = m = 0 + m$ **(satisfies iden)**

(3) For all $x \in P_m$ such that $x \neq 0$, $x + m$ and $m + x$ are undefined

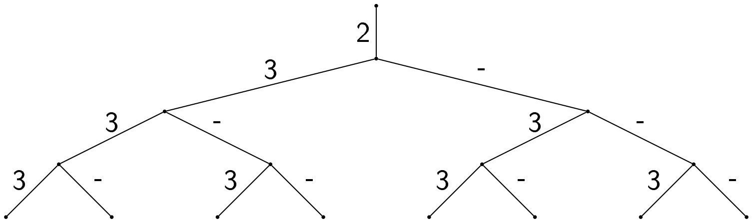
(4) For all $x, y \in P$ such that $x \oplus y$ is undefined, $x + y$ is not yet determined, which will be represented by $x + y = N$

Process of Filling out Table

$\oplus$	0	1	2
0	0	1	2
1	1	2	-
2	2	-	-

$\rightarrow$

$+$	0	1	2	3
0	0	1	2	3
1	1	2	N	-
2	2	N	N	-
3	3	-	-	-



Checking for cancellativity, conjugation and associativity

A table is **cancellative** if each element appears no more than once in every row/column.

Cancellative

+	0	1	2	3
0	0	1	2	3
1	1	2	-	-
2	2	-	3	-
3	3	-	-	-

Not Cancellative

+	0	1	2	3
0	0	1	2	3
1	1	2	3	-
2	2	-	3	-
3	3	-	-	-

$$1 + 2 = 3$$
$$2 + 2 = 3$$

Checking for Conjugation

A table is **conjugative** if for all i, j:

- each element defined in row i is also defined in column i
- each element defined in column j is also defined in row j.

Conjugative

+	0	1	2	3
0	0	1	2	3
1	1	2	3	-
2	2	3	-	-
3	3	-	-	-

Not Conjugative

+	0	1	2	3
0	0	1	2	3
1	1	2	3	-
2	2	-	-	-
3	3	-	-	-

$$1 + 2 = 3$$
$$\nexists u(u + 1 = 3)$$
$$\nexists v(2 + v = 3)$$

Checking for Associativity

A table is **associative** if for all $x, y, z \in P$:

- If $(x + y) + z$ is undefined, then $x + (y + z)$ is also undefined.
- If $(x + y) + z$ is defined, then $(x + y) + z = x + (y + z)$.

```
for all x,y,z in P_m where (x+y) is defined:
    if (x+y)+z is undefined:
        if y+z and x+(y+z) are defined:
            return False
    if (x+y)+z is defined:
        if y+z or x+(y+z) are undefined:
            return False
        if x+(y+z) != (x+y)+z:
            return False
return True
```

Counting generalized pseudo-effect algebras

n	O	PO	GO	GPO	E	PE	GE	GPE
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	0	0	1	1	1	1	2	2
4	1	1	2	2	3	3	5	5
5	0	1	2	3	4	5	12	13
6	1	2	4	7	10	12	35	42
7	0	2	8	19	14	19	119	171
8	2	5	18	68	40	52	496	1020
9	0	4	42		60	84		11742
10								322918

Table: Number of all algebras in each class

The Lean Theorem Prover: a brief introduction

Proofs about partial algebras and nonassociative algebras can be error-prone

Some equational or 1st-order proofs and small counterexamples can be found with automated theorem provers like Prover9/Mace4

For more advanced results an interactive higher-order theorem prover is useful

HOL, HOLlight, Coq, Isabelle have been developed over several decades

Lean is a fresh start by Leonardo de Moura at Microsoft (2013)

Aims to bridge the gap between interactive and automated theorem proving

Based on the Dependent Type Theory, constructive and classical reasoning

Proofs are checked as they are typed in Emacs or Visual Studio Code

Some Lean examples

Example: $\wedge$ elimination/introduction

```
variables (P Q : Prop)
theorem comm: P ∧ Q → Q ∧ P:=
assume L0: P ∧ Q,
  have L1: P, from and.left L0,
  have L2: Q, from and.right L0,
show Q ∧ P, from and.intro L2 L1
```

Let's see it live (there is even a browser version)

Some References

S. Alexander, P. Jipsen, N. Upegui: Generalizations of effect algebras as residuated partially ordered monoids, in progress

D. J. Foulis and M. K. Bennett: Effect algebras and unsharp quantum logics. Found. Phys. **24**, (1994), 1325–1346

F. A. Roumen: Effect algebroids, Ph. D. thesis, Radboud Universiteit, Nijmegen, 2017, viii+141

W. Rump and Y. C. Yang: Non-commutative logical algebras and algebraic quantales. Annals of Pure and Applied Logic, **165**, (2014), 759–785

Thanks!