The lattice of varieties generated by small residuated lattices

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Outline

- Lattice of finitely generated CD varieties
- The HS order on finite subdirectly irreducibles
- Computing finite residuated lattices
- Using automated theorem provers
- Amalgamation in residuated lattices

Residuated lattices – Substructural logics

A **residuated lattice** $(A, \vee, \wedge, \cdot, 1, \backslash, /)$ is an algebra where $(A, \vee, \wedge)$ is a **lattice**, $(A, \cdot, 1)$ is a **monoid** and for all $x, y, z \in A$

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y$$

Residuated lattices generalize many algebras related to **logic**, e. g. **Boolean algebras**, **Heyting algebras**, **MV-algebras**, Hajek's **basic logic algebras**, **linear logic algebras**, ...

FL = Full Lambek calculus = the starting point for substructural logics

corresponds to class FL of all residuated lattices **with a new constant** 0

Extensions of **FL** correspond to subvarieties of FL

Hiroakira Ono

(California, September 2006)

[1985] *Logics without the contraction rule*

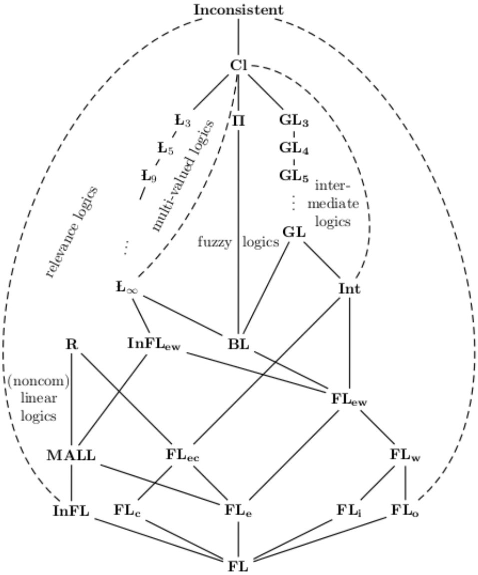
(with Y. Komori)

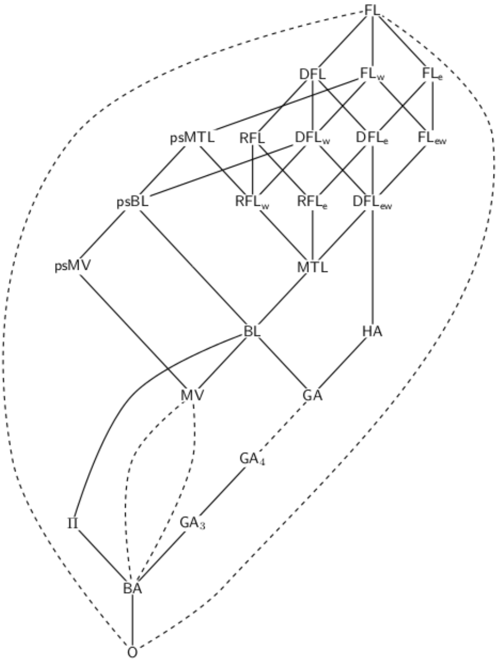
Provides a **framework** for studying many substructural logics, relating sequent calculi with semantics

The name **substructural logics** was suggested by K. Dozen, October 1990

[2007] *Residuated Lattices: An algebraic glimpse at substructural logics* (with Galatos, J., Kowalski)

Some propositional logics extending FL

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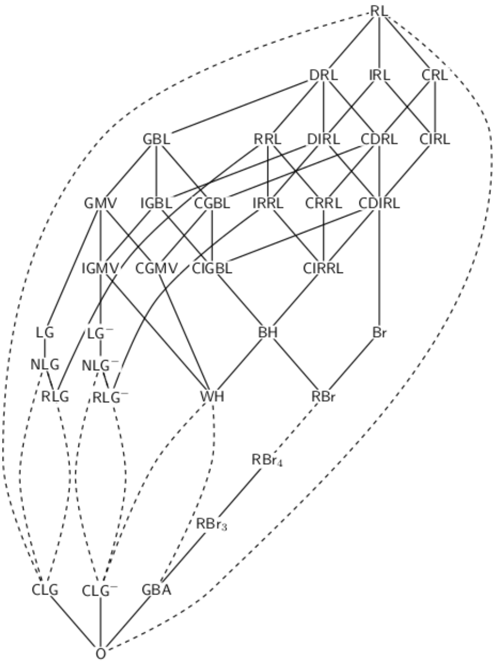
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Congruence distributive varieties

A class $\mathcal{V}$ of algebras is a **variety** if it is defined by **identities**

$\iff \quad \mathcal{V} = \mathbf{HSP}(\mathcal{K})$ for some class $\mathcal{K}$ of algebras

$\mathcal{V}$ is **finitely generated** if $\mathcal{K}$ can be a **finite** class of **finite algebras**

An algebra is **congruence distributive** (CD) if its lattice of congruences is distributive

A class $\mathcal{V}$ of algebras is CD if every member is CD

Who is this?

Bjarni Jónsson

(AMS-MAA meeting in Madison, WI 1968)

Algebras whose congruence lattices are distributive [1967]

* Jónsson's Lemma implies that the lattice of subvarieties of a CD variety is **distributive**
* The completely join-irreducibles in this lattice are generated by a **single s. i. algebra**
* for **finite** algebras A, B

$$\mathbf{HSP}\{A\} \subseteq \mathbf{HSP}\{B\} \iff A \in \mathbf{HS}\{B\}$$

The lattice of finitely generated varieties

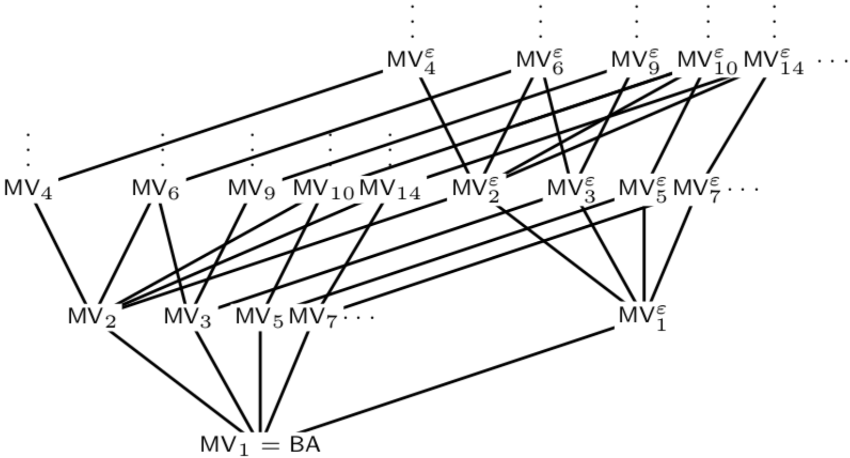
The relation $A \in \mathbf{HS}\{B\}$ is a **preorder** on algebras (since $\mathbf{SH} \leq \mathbf{HS}$)

For **finite s. i. algebras** in a CD variety it is a **partial order**

Called the HS-poset of the variety

The lattice of **finitely generated** subvarieties is given by **downsets** in this poset

The HS-poset of MV-algebras



Komori [1981] *Super-Lukasiewicz propositional logics*

Computing finite residuated lattices

First compute all **lattices** with n elements

[J. and N. Lawless 2013]: there are 1 901 910 625 578 for $n = 19$

Then compute all **lattice-ordered z-monoids** over each lattice

For **residuated lattices** there are 295292 for $n = 8$

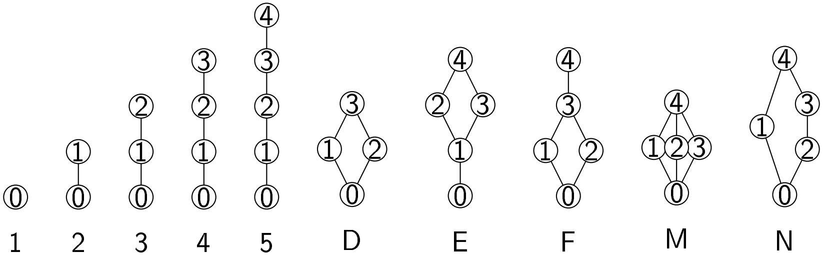
[Belohlavek and Vychodil 2010]: 30 653 419 CIRL of size $n = 12$

Remove **non-s. i. algebras** from list (very few)

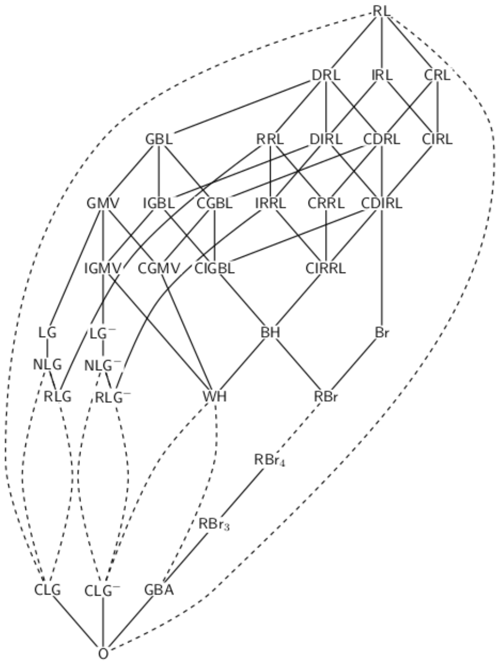
Compute maximal proper **subalgebras** of each algebra

Compute maximal **homomorphic images** (=minimal congruences)

A small sample



n	RL	Chn	DE	F	M	N	FL	Chn	DE	F	M	N
1	1	1					1	1				
2	1	1					2	2				
3	3	3					9	9				
4	20	15	5				79	60	19			
5	149	84	20	11	8	26	737	420	97	53	37	130
Tot	174	104					828	492				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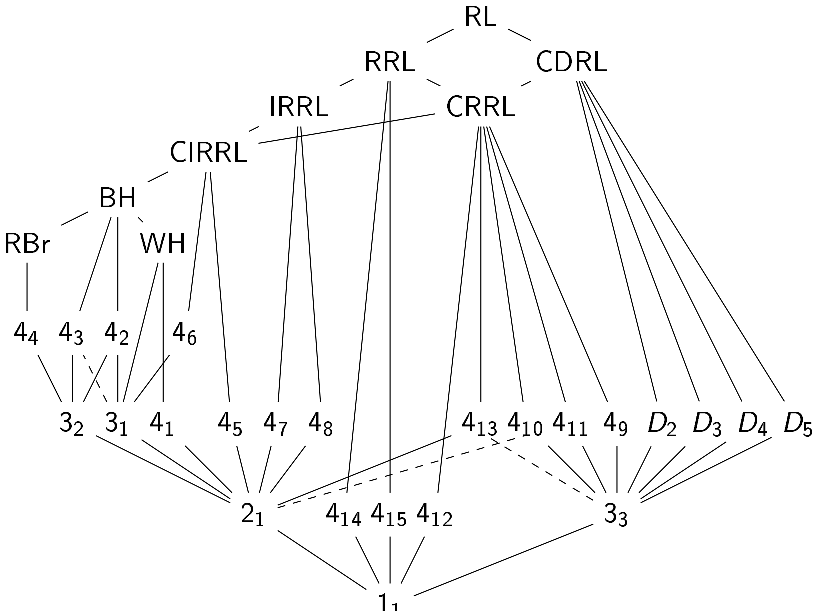
CIGBL
CIRRL
LG
LG$^-$
BH
Br
NLG
NLG$^-$
RLG
RLG$^-$
WH
RBr
RBr$_4$
RBr$_3$
CLG
CLG$^-$
GBA
O

Residuated lattices of size ≤ 4

RL var	FL var	Name, id, transformations	Sub	Hom
GBA	BA	$\langle 2_1, 1\rangle$		
WH	MV	$\langle 3_1, 2, 01\rangle$	2_1	
RBr	GA	$\langle 3_2, 2, 11\rangle$	2_1	2_1
CRRL	RInFL$_e$	$\langle 3_3, 1, 22\rangle$		
WH	MV	$\langle 4_1, 3, 001, 012\rangle$	2_1	
BH	BL	$\langle 4_2, 3, 011, 122\rangle$ $\langle 4_3, 3, 111, 112\rangle$	3_1 3_2 3_1 3_2	3_1 2_1
RBr	GA	$\langle 4_4, 3, 111, 122\rangle$	3_2	3_2
CIRRL	RInFL$_{ew}$	$\langle 4_5, 3, 001, 022\rangle$	2_1	2_1
CIRRL	RFL$_{ew}$	$\langle 4_6, 3, 001, 002\rangle$	3_1	
IRRL	RFL$_w$	$\langle 4_7, 3, 001, 122\rangle$ $\langle 4_8, 3, 011, 022\rangle$	2_1 2_1	
CRRL	RInFL$_e$	$\langle 4_9, 1, 233, 333\rangle$ $\langle 4_{10}, 2, 113, 333\rangle$	3_3 2_1 3_3	 3_3
CRRL	RFL$_e$	$\langle 4_{11}, 1, 223, 333\rangle$ $\langle 4_{12}, 2, 011, 133\rangle$ $\langle 4_{13}, 2, 111, 133\rangle$	3_3 3_3	 2_1
RRL	RFL	$\langle 4_{14}, 2, 111, 333\rangle$ $\langle 4_{15}, 2, 113, 133\rangle$		
GBA	BA	$\langle D_1, 3, 101, 022\rangle$	2_1	2_1
CDRL	DInFL$_e$	$\langle D_{2,1}, 1, 202, 323\rangle$ $\langle D_{3,1}, 1, 213, 333\rangle$ $\langle D_{4,2}, 1, 233, 333\rangle$	3_3 3_3 3_3	
CDRL	DFL$_e$	$\langle D_5, 1, 222, 323\rangle$	3_3	

HS-poset of residuated lattices with $\leq$ 4 elements



The Amalgamation Property

Let $\mathcal{K}$ be a class of mathematical structures

(e. g. sets, groups, residuated lattices, ...)

Usually there is a natural notion of morphism for $\mathcal{K}$

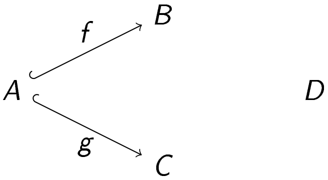
(e. g. function, homomorphism,...)

$\mathcal{K}$ has the **amalgamation property** if

for all $A, B, C \in \mathcal{K}$ and all **injective** $f : A \hookrightarrow B$, $g : A \hookrightarrow C$

there exists $D \in \mathcal{K}$ and **injective** $h : B \hookrightarrow D$, $k : C \hookrightarrow D$ such that

$h \circ f = k \circ g$



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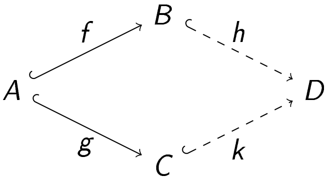
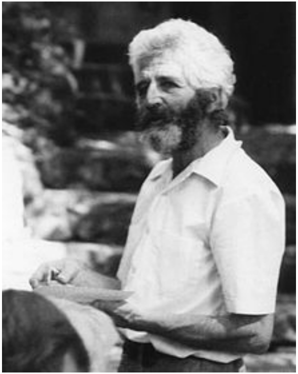


Connections with logic



Bill Craig

(Berkeley, CA 1977)

Craig interpolation theorem [1957]

If $\phi \implies \psi$ is true in first order logic

then there exists θ containing only

the relation symbols in both ϕ, ψ

such that $\phi \implies \theta$ and $\theta \implies \psi$

Also true for many other logics, including classical propositional logic and intuitionistic propositional logic

Let $\mathcal{K}$ be a class of algebras of an algebraizable logic $\mathcal{L}$

Then $\mathcal{K}$ has the (strong/super) amalgamation property iff $\mathcal{L}$ satisfies the Craig interpolation property

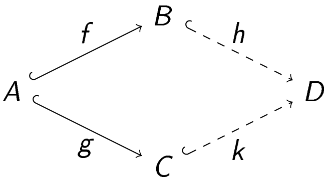
What is known?

There are two versions: 1. the **amalgamation property** (AP)

for all $A, B, C \in \mathcal{K}$ and all **injective** $f : A \hookrightarrow B$, $g : A \hookrightarrow C$

there exists $D \in \mathcal{K}$ and **injective** $h : B \hookrightarrow D$, $k : C \hookrightarrow D$ such that

$h \circ f = k \circ g$



2. the **strong amalgamation property** (SAP): in addition to

$h \circ f = k \circ g$ also require $h[f[A]] = h[B] \cap k[C]$

Equivalently: If A is a subalgebra of B, C in $\mathcal{K}$ and $A = B \cap C$ then there exists $D \in \mathcal{K}$ such that B, C are subalgebras of D

A sample of what is known

These categories **have** the **strong amalgamation property**:

Sets
Groups [Schreier 1927]
Sets with any binary operation [Jónsson 1956]
Variety of all algebras of a fixed signature
Partially ordered sets [Jónsson 1956]
Lattices [Jónsson 1956]

These categories **only** have the **amalgamation property**:

Distributive lattices [Pierce 1968]
Abelian lattice-ordered groups [Pierce 1972]

These categories **fail** to have the **amalgamation property**:

Semigroups [Kimura 1957]
Lattice-ordered groups [Pierce 1972]

Kiss, Márki, Pröhle and Tholen [1983] Categorical algebraic properties. A **compendium on amalgamation**, congruence extension, epimorphisms, residual smallness and injectivity

They summarize some general techniques for establishing these properties

They give a table with **known results for 100 categories**

Day and Jezek [1984] The **only** lattice varieties that satisfy **AP** are the **trivial variety**, the **variety of distributive lattices** and **the variety of all lattices**

Busianiche and Montagna [2011]: *Amalgamation, interpolation and Beth's property in* **BL** (Section 6 in Handbook of Mathematical Fuzzy Logic)

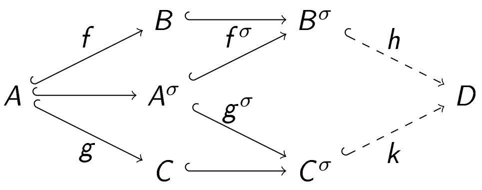
Metcalfe, Montagna and Tsinakis [2014]: *Amalgamation and interpolation in ordered algebras*, Journal of Algebra

How to prove/disprove the AP

Look at **three** examples:

1. Why does **SAP hold** for class of all **Boolean algebras**?
2. Why does **AP hold** for **distributive lattices**?
3. Why does **AP fail** for **distributive residuated lattices**?

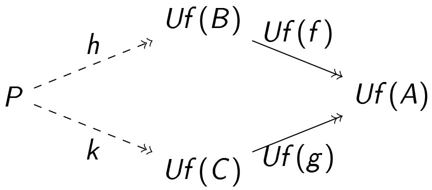
1. Boolean algebras (**BA**) can be embedded in **complete and atomic** Boolean algebras (**caBA**)



caBA is dually equivalent to **Set**

Amalgamation for BA

So we need to fill in the following dual diagram in **Set**



Can take P to be the **pullback**, so
$P = \{(b, c) \in Uf(B) \times Uf(C) : Uf(f)(b) = Uf(g)(c)\}$

Then $h = \pi_1|_P$ and $k = \pi_2|_P$

h is **surjective** since for all $b \in Uf(B)$, there exists $c \in Uf(C)$ s.t. $Uf(f)(b) = Uf(g)(c)$ because $Uf(g)$ is **surjective**

Similarly k is **surjective**

2. Amalgamation for distributive lattices

Theorem [J. and Rose 1989]: Let $\mathcal{V}$ be a **congruence distributive** variety whose members have **one-element** subalgebras, and assume that $\mathcal{V}$ is generated by a **finite simple algebra** that has **no proper nontrivial subalgebras**. Then $\mathcal{V}$ has the **amalgamation property**.

The variety of distributive lattices is generated by the **two-element lattice**, which is **simple** and has only **trivial proper subalgebras**, hence **AP holds**.

Corollary: The **AP holds** for the variety of **Sugihara algebras** ($= V(3_3)$), and for $V(4_{12})$, $V(4_{14})$, $V(4_{15})$ as well as for any variety generated by an atom of the HS-poset

3. AP fails for distributive residuated lattices

Finally we get to mention some **computational tools**

To **disprove AP** or **SAP**, we essentially want to search for 3 **small** models A, B, C in $\mathcal{K}$ such that A is a **submodel** of both B and C

We use the **Mace4 model finder** from **Bill McCune [2009]** to enumerate nonisomorphic models $A_1, A_2, \ldots$ in a **finitely axiomatized** first-order theory Σ

For each A_i we construct the **diagram** Δ_i and use **Mace4** again to find all **nonisomorphic** models $B_1, B_2, \ldots$ of $\Delta_i \cup \Sigma \cup \{\neg(c_a = c_b) : a \neq b \in A_i\}$ with **slightly more** elements than A_i

Note that **by construction**, each B_j has A_i as submodel

Checking failure of AP

Iterate over **distinct** pairs of models B_j, B_k and construct the theory Γ that extends Σ with the **diagrams of these two models**, using only **one set of constants** for the overlapping submodel A_i

Add formulas to Γ that ensure all constants of B_j are **distinct**, and same for B_k

Use **Mace4** to check for a **limited** time whether Γ is satisfiable in some **small** model

If not, use the **Prover9 automated theorem prover** (McCune [2009]) to search for a proof that Γ is **inconsistent**. If **yes**, then a **failure of AP** has been found

To check if **SAP fails**, add formulas that ensure constants of **each pair** of models **cannot** be identified, and **also iterate** over pairs B_j, B_j

Amalgamation for residuated lattices

Open problem: Does **AP** hold for all **residuated lattices**?

Commutative residuated lattices satisfy $x \cdot y = y \cdot x$

Kowalski, Takamura ['04] **AP** holds for commutative resid. lattices

Distributive residuated lattices satisfy $x \wedge (y \wedge z) = (x \wedge y) \vee (x \wedge z)$

Theorem [J. 2014]: AP fails for any variety of distributive residuated lattices that includes two specific 5-element commutative distributive integral residuated lattices

In particular, **AP fails** for the varieties DRL, CDRL, IDRL, CDIRL and any varieties between these

Conclusion

Many other **minimal** failures of **AP** and **SAP** can be found automatically

By studying the **amalgamations of small algebras** one can get **hints** of how **AP** may be proved in general

The method of enumerating small models and using **diagrams** of structures in **automated theorem provers** is applicable to many other problems, e.g., in coalgebra, combinatorics, finite model theory, ...

Computational research tools like **Sage**, **Prover9**, **UACalc**, **Isabelle**, **Coq**, ... are becoming **very useful** for research in algebra, logic and combinatorics

Thanks!