Partially-ordered multi-type algebras, multi-type frames and the category of polarities

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joint work with
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Overview

- Partially-ordered multi-type algebras
- Semi-De Morgan algebras
- Multi-type frame semantics
- Term-equivalence for multi-type algebras

A series of papers about cut-free display calculi

Dynamic epistemic logic displayed [Greco, Kurz, Palmigiano, 2013]

Linear logic properly displayed [Greco and Palmigiano, 2016]

Lattice logic properly displayed [Greco and Palmigiano, 2016]

Bilattice logic properly displayed [Greco, Liang, Palmigiano and Rivieccio, 2017]

Multi-type display calculus for semi-DeMorgan logic [Greco, Liang, Moshier and Palmigiano, 2017]

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A **multi-type algebra** is of the form $\mathbb{A} = ((A_\tau)_{\tau\in\mathscr{T}}, \mathscr{F})$ where each $f \in \mathscr{F}$ is a function $f : A_{\tau_1} \times \cdots \times A_{\tau_n} \to A_\tau$ for some $\tau_1, \ldots, \tau_n, \tau \in \mathscr{T}$.

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They have applications, e.g., in **algebraic logic** and as **abstract data types** in computer science.

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Varieties and **quasivarieties** of po-algebras were studied by Pigozzi [2004]

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The decomposition of lattice-ordered unitype algebras into simpler loosely connected multi-type components can lead to decision procedures for the equational theory, using cut-free sequent calculi.

An example: Semi-De Morgan algebras

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DMA is easily seen to be canonical, i.e., closed under canonical extensions

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$(b \sqcup c)^- = h((e(h((e(b) \vee e(c))'')))') = h((b \vee c)''') = (b \vee c)' = b' \wedge c'$

$b^- \sqcap c^- = h(e(h((e(b))')) \wedge e(h((e(c))'))) = h((e(b))') \wedge h((e(c))') =$
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With the standard definition of multi-type homomorphism, **F** and **G** are functors that give an equivalence of categories.

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Hence semi-DeMorgan algebras are **canonical**.

Furthermore, [Greco, Liang, Moshier, Palmigiano 2017] provide a **cut-free multi-type display calculus** for HSM-algebras that has the **sub-formula property**.

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DML is easily seen to be canonical, i.e., closed under canonical extensions

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Again a **categorical equivalence** can be established between semi-DeMorgan lattices and HSM-lattices.

For another example, a **linear logic algebra with exponentials** decomposes into a commutative residuated lattice and a Heyting algebra connected by appropriate maps, which leads to a cut-free display calculus for linear logic [Greco and Palmigiano 2016].

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3. implies $x \cdot y = \gamma_N(x \circ y)$ is residuated on $\mathbf{W}^+ = (\gamma_N[\mathscr{P}(W)], \vee, \cap, \cdot, \backslash, /, 1)$
where $\gamma_N(X) = N^\downarrow N^\uparrow(X)$, $X\backslash Z = \{y \in W : X \circ \{y\} \subseteq Z\}$, $1 = \gamma_N(E)$.

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Fortunately Drew Moshier has provided the appropriate notion of morphism for polarities.

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Complete lattices with complete homomorphisms form a category

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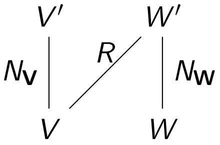
In either case we say that R is **compatible**



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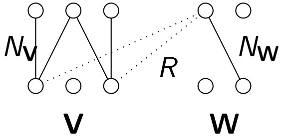

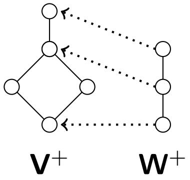

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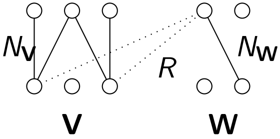

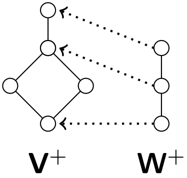

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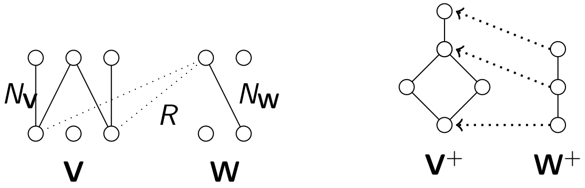

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$\implies$ The category of frames with compatible morphisms is **equivalent** to the category of complete lattices with completely join-preserving maps.

Lattice compatible morphisms

Lemma: $R^{\downarrow}N^{\uparrow}_{\mathbf{W}}$ preserves $\bigvee$ iff there exists a compatible relation $R_*: W \to V'$ such that $R^{\downarrow}N^{\uparrow}_{\mathbf{W}} = N^{\downarrow}_{\mathbf{V}}R^{\uparrow}_*$ (call R lattice compatible)

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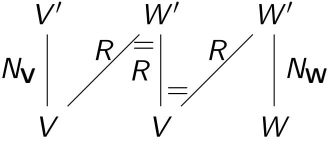
Note that every morphism has **itself** as epi-mono factorization



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Use this framework to design a cut-free Gentzen system with the subformula property and check if this gives a decision procedure for equations of semi-DeMorgan lattices.

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A **multi-type algebra** for $\mathscr{C}$ is a product preserving functor $\mathbf{A} : \mathscr{C} \to \mathbf{Set}$ (the category of sets). For any morphism f, the function $\mathbf{A}f$ is an operation of the algebra.

Lawvere theories for po-multi-type algebras

A **pom-algebra** for $\mathscr{C}$ is a product preserving functor $\mathbf{A} : \mathscr{C}_\partial \to \mathbf{Pos}_\partial$ (the category of posets with order-preserving maps expanded with a natural transformation $\partial_P : P \to P^\partial$ given by $\partial_P(x) = x$, and similarly for $\mathscr{C}_\partial$).

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This is another tool to adjust the presentation of a logic (or class of algebras) to possibly make it easier to work with.

Some references

Birkhoff G., Lipson J. D. (1970) Heterogeneous Algebras, Journal of Combinatorial Theory, **8**, 115–133

Caleiro C., Gonçalves R. (2007) On the Algebraization of Many-Sorted Logics. WADT 2006. LNCS, vol 4409. Springer, Berlin, Heidelberg

Frittella S., Greco G., Kurz A., Palmigiano A., Sikimi\'c V. (2016) Multi-type sequent calculi, https://arxiv.org/abs/1609.05343

Galatos N., Jipsen P. (2013) Residuated frames with applications to decidability, Transactions of the American Mathematical Society **365**(3) 1219–1249

Greco G., Kurz A., Palmigiano A. (2013) Dynamic epistemic logic displayed. In: Grossi D., Roy O., Huang H. (eds) Logic, Rationality, and Interaction. LORI 2013. LNCS, vol 8196. Springer

Greco G., Liang F., Moshier M.A., Palmigiano A. (2017) Multi-type display calculus for semi De Morgan logic. WoLLIC 2017. LNCS, vol 10388. Springer

Greco G., Liang F., Palmigiano A., Rivieccio U. (2017) Bilattice logic properly displayed, https://arxiv.org/abs/1709.02167

Greco G. and Palmigiano A. Linear logic properly displayed, https://arxiv.org/abs/1611.04181

Greco G. and Palmigiano A. Lattice logic properly displayed, https://arxiv.org/abs/1612.05930

Pigozzi D. (2004) Partially ordered varieties and quasivarieties,

https://orion.math.iastate.edu/dpigozzi/notes/santiago_notes.pdf

Thanks!