Residuated lattices, Kripke semantics and correspondence

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Outline

- Residuated lattices
- Contexts and their morphisms
- Completions and canonical extension
- Extending Urquhart and Hartung's duality for lattices
- Residuated frames and their morphisms
- Correspondence
- Open problems

Residuated lattices

A **residuated lattice** is an algebra $\mathbb{A} = (A, \vee, \wedge, \cdot, \backslash, /, 1)$ such that $(A, \vee, \wedge)$ is a **lattice**, $(A, \cdot, 1)$ is a **monoid** and for all $x, y, z \in A$

$$xy \leq z \iff y \leq x\backslash z \iff x \leq z/y$$

$\mathbb{A}$ is **integral** if it satisfies $x \leq 1$ i.e. $x \wedge 1 = x$

$\mathbb{A}$ is **divisible** if it satisfies $x \leq y \implies x = y(y\backslash x) = (x/y)y$

$\mathbb{A}$ is **commutative** if it satisfies $xy = yx$ hence $y\backslash x = x/y$

$\mathbb{A}$ is **modular** if it satisfies $x \leq z \implies (x \vee y) \wedge z = x \vee (y \wedge z)$

$\mathbb{A}$ is **distributive** if it satisfies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$\mathbb{A}$ is **bounded** if it has a constant $\perp$ that satisfies $\perp \leq x$

FL-algebras

A **Full Lambek algebra** (or FL-algebra) is a residuated lattice with an additional **constant** 0

No properties are assumed about this constant

An FL_w-algebra satisfies **weakening**: $0 \leq x \leq 1$

An FL_e-algebra is a **commutative** FL-algebra (e for **exchange**)

An FL_c-algebra is **contractive** or **square-increasing** if $x \leq xx$

Linear negations are defined as $\sim x = x\backslash 0$ and $-x = 0/x$

An **involutive** FL-algebra (InFL-algebra) satisfies $\sim -x = x = -\sim x$

A **cyclic** FL-algebra (CyFL-algebra) satisfies $\sim x = -x$

A **Heyting algebra** is an FL-algebra that satisfies $xy = x \wedge y$

Substructural logic

Residuated lattices and **FL-algebras** are the **algebraic semantics** of **substructural logics**

Gentzen defined **sequent proof systems** LK and LJ for **classical** and **intuitionistic logic**

Omitting (some of) the **structural** rules like **weakening** $\frac{\varphi \Rightarrow \psi}{\varphi \cdot \theta \Rightarrow \psi}$, **exchange** $\frac{\varphi \cdot \theta \Rightarrow \psi}{\theta \cdot \varphi \Rightarrow \psi}$ and **contraction** $\frac{\varphi \cdot \varphi \Rightarrow \psi}{\varphi \Rightarrow \psi}$ gives proof systems for **substructural logic**

From a sequent system one can construct **relational semantics**, leading to **residuated frames**

Start with the logical rules to find relational semantics for lattices

Contexts (or polarities)

A ***context*** is a structure $\mathbb{W} = (W, W', I_{\mathbb{W}})$ such that

W, W' are sets (of states and co-states) and $I_{\mathbb{W}} \subseteq W \times W'$

The ***incidence relation*** $I_{\mathbb{W}}$ determines two functions
$I^{\uparrow} : \mathcal{P}(W) \to \mathcal{P}(W')$ and $I^{\downarrow} : \mathcal{P}(W') \to \mathcal{P}(W)$ by

$I^{\uparrow}A = \{b : \forall a \in A\ aIb\}$ and $I^{\downarrow}B = \{a : \forall b \in B\ aIb\}$

Gives a ***Galois connection*** from $\mathcal{P}(W)$ to $\mathcal{P}(W')$, i.e.,
$A \subseteq I^{\downarrow}B \iff B \subseteq I^{\uparrow}A$ for all $A \subseteq W$ and $B \subseteq W'$
The **Galois-closure** of $A \subseteq W$ is $\gamma A = I^{\downarrow}I^{\uparrow}A$

$\mathbb{W}^{+} = (\{\gamma A : A \subseteq W\}, \bigcap, \bigvee)$ and $\mathbb{W}'^{+} = (\{\gamma' B : B \subseteq W'\}, \bigcap, \bigvee)$
are **dually isomorphic complete lattices** with **intersection as meet** and **Galois-closure of union as join**.

Examples

Contexts are due to Birkhoff; studied in Formal Concept Analysis

Let L be a (bounded) lattice

The **Dedekind-MacNeille context** is $DM(L) = (L, L, \leq)$

$DM(L)^+$ is the **MacNeille completion** of L

Let $\mathcal{F}L = \{\text{nonempty filters of } L\}$, $\mathcal{I}L = \{\text{nonempty ideals of } L\}$

Define $I \subseteq \mathcal{F}L \times \mathcal{I}L$ by $\qquad F\ I\ D$ iff $F \cap D \neq \emptyset$

The **filter-ideal context** is $FI(L) = (\mathcal{F}L, \mathcal{I}L, I)$

$FI(L)^+$ is the **canonical extension** L^δ of L

Completions

A **completion** of a lattice L is an embedding $e : L \to \bar{L}$ where $\bar{L}$ is a complete lattice

The MacNeille completion is **join-dense** and **meet-dense** (every $x \in DM(L)^+$ is a **join** of elements of L and a **meet** of elements of L)

$\implies$ all existing joins and meets of L are **preserved**

The **canonical extension** is **dense** (every $x \in L^\delta$ is a **join of a meet** of elements of L and a **meet of a join** of elements of L)

and **compact** (if $\bigwedge A \leq \bigvee B$ for $A, B \subseteq L$ then $a_1 \wedge \cdots \wedge a_m \leq b_1 \vee \cdots \vee b_m$ for some $a_i \in A, b_j \in B$)

$\implies$ all infinite joins and meets of L are destroyed

So these contexts are two extremes of lattice completions

Contexts in general are a good way to study dense completions

Context morphisms

Complete lattices with complete homomorphisms form a category

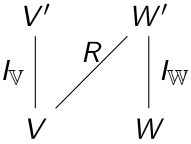
What are the **appropriate** morphisms for contexts?

For a context $\mathbb{W} = (W, W', I)$ the relation I is an identity morphism that induces the identity map $\gamma = I^{\downarrow} I^{\uparrow}$ on the closed sets

A **context morphism** $R : \mathbb{V} \to \mathbb{W} = (V, W', R)$ is a relation $R \subseteq V \times W'$ such that $I_{\mathbb{V}}^{\downarrow} I_{\mathbb{V}}^{\uparrow} R^{\downarrow} = R^{\downarrow} = R^{\downarrow} I_{\mathbb{W}}^{\uparrow} I_{\mathbb{W}}^{\downarrow}$

or equivalently $R^{\uparrow} I_{\mathbb{V}}^{\downarrow} I_{\mathbb{V}}^{\uparrow} = R^{\uparrow} = I_{\mathbb{W}}^{\uparrow} I_{\mathbb{W}}^{\downarrow} R^{\uparrow}$

In either case we say that R is compatible



Compatible morphisms $\equiv^\partial$ meet-semilattice homomorphisms

Lemma. *If R is compatible then $R^{\downarrow}I_{\mathbb{W}}^{\uparrow}:\mathbb{W}^+\to\mathbb{V}^+$ preserves $\bigcap$*

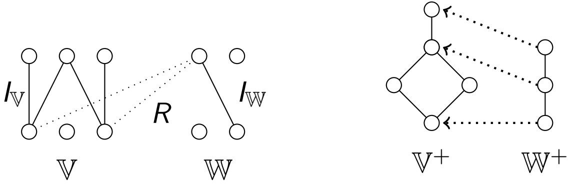
Proof: Let $\{A_k : k\in K\}$ be a family of Galois closed sets of W

Since $R^{\downarrow}I_{\mathbb{W}}^{\uparrow}I_{\mathbb{W}}^{\downarrow}=R^{\downarrow}$,

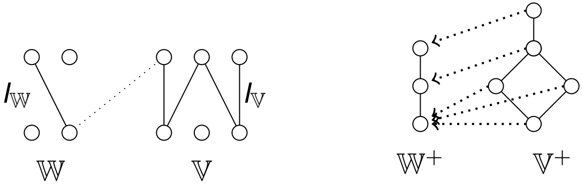
$R^{\downarrow}I_{\mathbb{W}}^{\uparrow}\bigcap A_k = R^{\downarrow}\bigvee I_{\mathbb{W}}^{\uparrow}A_k = R^{\downarrow}I_{\mathbb{W}}^{\uparrow}I_{\mathbb{W}}^{\downarrow}\bigcup I_{\mathbb{W}}^{\uparrow}A_k = R^{\downarrow}\bigcup I_{\mathbb{W}}^{\uparrow}A_k$

$=\bigcap R^{\downarrow}I_{\mathbb{W}}^{\uparrow}A_k$ because $R^{\downarrow}\bigcup B_k=\bigcap R^{\downarrow}B_k$ always

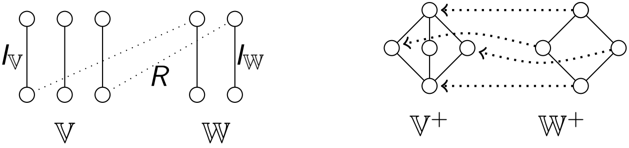
The result is in $\mathbb{V}^+$ since $R^{\downarrow}=I_{\mathbb{V}}^{\downarrow}I_{\mathbb{V}}^{\uparrow}R^{\downarrow}$ $\square$



More examples



Urquhart, Hartung and Suzuki only define duals of surjective lattice homomorphims. Here is the dual of a non-surjective homomorphism:

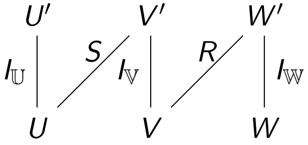


The category of contexts

Theorem. [Moshier 2012] (i) *The collection* ***Cxt*** *of all contexts with compatible relations as morphisms is a category*

If $S : \mathbb{U} \to \mathbb{V}$ *and* $R : \mathbb{V} \to \mathbb{W}$ *then* ***composition*** *is given by*

$xS\,\underline{;}\,Ry$ *iff* $x \in S^{\downarrow} I_{\mathbb{V}}^{\uparrow} R^{\downarrow}\{y\}$



(ii) *The category* ***Cxt*** *is dual to the category* ***INF*** *of complete semilattices with completely meet-preserving homomorphisms*

The adjoint functors are $^+ : \mathbf{Cxt} \to \mathbf{INF}$ *and* $\mathrm{DM} : \mathbf{INF} \to \mathbf{Cxt}$

On morphisms, $R^+ = R^{\downarrow} I_{\mathbb{W}}^{\uparrow} : \mathbb{W}^+ \to \mathbb{V}^+$ *and for an*

INF *morphism* $h : L \to M$, $\mathrm{DM}(h) = \{(x, y) \in M \times L : x \le h(y)\}$

Lattice compatible morphisms

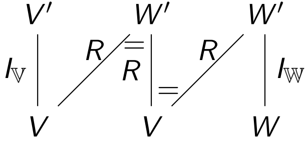
Lemma $R^{\downarrow} I^{\uparrow}_{\mathbb{W}}$ *preserves* $\bigvee$ *iff there exists a compatible relation* $R_* : W \to V'$ *such that* $R^{\downarrow} I^{\uparrow}_{\mathbb{W}} = I^{\downarrow}_{\mathbb{V}} R^{\uparrow}_*$ (call R lattice compatible)

Theorem *The category* **LCxt** *of all contexts with lattice compatible relations as morphisms is dual to the category* **CLat** *of complete lattices with complete lattice morphisms.*

Lemma *(i)* $R : \mathbb{V} \to \mathbb{W}$ *is a monomorphism in* **Cxt** *iff* $R^{\downarrow} R^{\uparrow} = I^{\downarrow}_{\mathbb{V}} I^{\uparrow}_{\mathbb{V}}$

(ii) $R : \mathbb{V} \to \mathbb{W}$ *is a epimorphism in* **Cxt** *iff* $R^{\uparrow} R^{\downarrow} = I^{\uparrow}_{\mathbb{W}} I^{\downarrow}_{\mathbb{W}}$

Note that every morphism has **itself** as epi-mono factorization



Perfect lattices and reduced contexts

LCxt is "much larger" than **CLat** since many different (but isomorphic) contexts represent the same lattice

$J(L) = \{j \in L : j \text{ is completely join-irreducible}\}$
$M(L) = \{m \in L : m \text{ is completely meet-irreducible}\}$

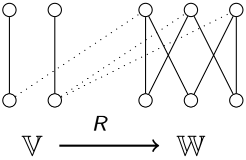
A lattice is **perfect** if every element is a **join of completely join-irreducibles** and a **meet of completely meet-irreducibles**
E.g. a Boolean algebra is **perfect** iff it is **atomic**

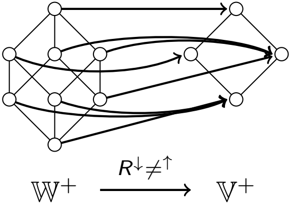
For a finite lattice L the **reduced context** $(J(L), M(L), \leq)$ is isomorphic to $DM(L)$

For finite L the reduced context can be **logarithmic** in size of L

E.g. a finite BA B has $(At(B), coAt(B), \leq) \cong (At(B), At(B), \neq)$ as reduced context

Morphisms between Boolean contexts





On complete and atomic Boolean algebras, any relation is a **Cxt** morphism

So there are $2^{2\cdot 3} = 64$ meet-homomorphisms from $\mathbb{V}$ to $\mathbb{W}$

For the example above, only six relations are **LCxt** morphisms

Overview of Dualities

Algebras w. homs		Spaces w. "cont" maps
CABool	$\equiv^\partial$	**Sets**
Bool	$\equiv^\partial$	**Stone**
CPerfDLat	$\equiv^\partial$	**Poset**
BDLat	$\equiv^\partial$	**Priestley** or **Spectral**
Spatial Frames	$\equiv^\partial$	Sober Spaces
INF or **SUP**	$\equiv^\partial$	**Cxt** (Moshier '12)
CLat	$\equiv^\partial$	**LCxt**
$\mathbf{JSLat}_0$	$\equiv^\partial$	**ACxt**
Lat w. surj. homs	$\equiv^\partial$	Urquhart 78/Hartung 92

Aim: Extend Hartung's duality so it works for **all** lattice homomorphism and for **all** contexts

Algebraic Contexts

$\mathbb{W} = (W, W', I)$ is an **algebraic context** if $I^{\downarrow}I^{\uparrow}A = \bigcup\{I^{\downarrow}I^{\uparrow}F : F \subseteq_\omega A\}$ for all $A \subseteq W$

$\iff$ $I^{\downarrow}I^{\uparrow}$ preserves directed unions

ACxt is the class of all **algebraic contexts**

$R : \mathbb{V} \to \mathbb{W}$ is an **ACxt morphism** if $R^{\downarrow}I^{\uparrow}_{\mathbb{W}}$ preserves directed unions

The **compact sets** are $Comp(\mathbb{W}) = \{I^{\downarrow}I^{\uparrow}F : F \subseteq_\omega W\}$

Theorem *The category* $\mathbf{JSLat}_0$ *is dual to* **ACxt**. *The adjoint functors are* $Comp : \mathbf{ACxt} \to \mathbf{JSLat}_0$ *and* $C_{\mathcal{I}} : \mathbf{JSLat}_0 \to \mathbf{ACxt}$, *where* $C_{\mathcal{I}}(L) = (L, \mathcal{I}L, I)$ *and* $I = \{(a, D) : a \in D\}$.

On morphisms, $Comp(R) = R^{\downarrow}I^{\uparrow} : Comp(Y) \to Comp(X)$ *and for a* $\mathbf{JSLat}_0$ *morph.* $h : L \to M$,
$C_{\mathcal{I}}(h) = \{(a, D) \in M \times \mathcal{I}L : h^{-1}[a] \cap D \neq \emptyset\}$

Topological Contexts

$\mathbb{W} = (W, W', I)$ is a **topological context** if

(i) W, W' are topological spaces with **closed** topologies τ, τ'

(ii) $\gamma[\tau] \subseteq \tau$ and $\gamma'[\tau'] \subseteq \tau'$

(iii) $L(\mathbb{W}) = \{A \in \tau : \gamma(A) = A \text{ and } I^{\uparrow}A \in \tau'\}$ is **dense** and **compact**

(iv) $L(\mathbb{W})$ is a subbasis of closed sets for τ
$I^{\uparrow}[L(\mathbb{W})]$ is a subbasis of closed sets for τ'

(v) If $\mathcal{D}$ is a $\subseteq$-directed subset of $L(\mathbb{W})$ then $\bigcup \mathcal{D}$ is Galois closed

Lemma $L(\mathbb{W})$ is a lattice, meet $= \cap$ and join $=$ Galois-closure of $\cup$

$R : \mathbb{V} \to \mathbb{W}$ is a **TCxt** morphism if it is a **LCxt** morphism and $R^{\downarrow}I^{\uparrow}|_{L(W)}$ maps into $L(\mathbb{W})$

For a bounded lattice $\mathbb{L}$ the **topological context of** $\mathbb{L}$ is $C(\mathbb{L}) = (\mathcal{F}\mathbb{L}, \mathcal{I}\mathbb{L}, I)$ where $F\ I\ D$ iff $F \cap D \neq \emptyset$

Let $\mathcal{F}_a = \{F \in \mathcal{F}\mathbb{L} : a \in F\}$ and $\mathcal{D}_a = \{D \in \mathcal{I}\mathbb{L} : a \in D\}$

τ has a closed subbasis $\{\mathcal{F}_a : a \in \mathbb{L}\}$, τ' has a closed subbasis $\{\mathcal{D}_a : a \in \mathbb{L}\}$

Hartung '92 shows $L(C(\mathbb{L})) = \{\mathcal{F}_a : a \in \mathbb{L}\}$

Theorem *The category* **BLat** *of bounded lattices with 0, 1-preserving lattice homomorphisms is dual to* **TCxt**. *The adjoint functors are* L : **TCxt** $\to$ **BLat** *and* C : **BLat** $\to$ **TCxt**.

$L(R) = R^{\downarrow}I^{\uparrow} : L(\mathbb{W}) \to L(\mathbb{V})$ *and for* $h : \mathbb{L} \to \mathbb{M}$,
$C(h) = \{(F, D) \in \mathcal{F}\mathbb{M} \times \mathcal{I}\mathbb{L} : h^{-1}[F] \cap D \neq \emptyset\}$

Some details

Lemma. If h is a function and F, D are subsets of its domain and codomain respectively then $h^{-1}[F] \cap D \neq \emptyset$ if and only if $F \cap h[D] \neq \emptyset$.

Lemma. L : **TCxt** $\to$**Lat** and C : **Lat** $\to$ **TCxt** are functors.

Proof. Let $\mathbb{U}, \mathbb{V}, \mathbb{W}$ be topological contexts and let $\mathbb{L}, \mathbb{M}, \mathbb{N}$ be bounded lattices.

$L(\mathbb{W})$ is a bounded lattice under finite intersections and Galois-closure of finite unions.

Likewise $C(\mathbb{L})$ is a topological context by the results of Hartung 1992 and Gehrke-Harding 2001.

Suppose $h : \mathbb{L} \to \mathbb{M}$ is a bounded lattice homomorphism.

We need to check that Ch is a **TCxt** morphism from $C(\mathbb{M}) = (\mathcal{F}\mathbb{M}, \mathcal{I}\mathbb{M}, I)$ to $C(\mathbb{L}) = (\mathcal{F}\mathbb{L}, \mathcal{I}\mathbb{L}, I)$.

Ch is compatible:
$Ch^{\downarrow}\{D\} = \{F \in \mathcal{F}\mathbb{L} : F \cap h[D] \neq \emptyset\} = I^{\downarrow}\{\downarrow h[D]\} = I^{\downarrow}I^{\uparrow}Ch^{\downarrow}\{D\}$

$Ch^{\uparrow}\{F\} = \{D \in \mathcal{I}L : h^{-1}[F] \cap D \neq \emptyset\} = I^{\uparrow}\{h^{-1}[F]\} =$
$I^{\uparrow}I^{\downarrow}Ch^{\uparrow}\{F\}$

Counting finite reduced separating contexts

A context is **separating** if $x \neq y$ in W implies $\gamma(x) \neq \gamma(y)$ and $x \neq y$ in W' implies $\gamma'(x) \neq \gamma'(y)$

It is **reduced** if $\gamma(\gamma(x) - \{x\}) \neq \gamma(x)$ and same for γ'

For any perfect lattice $(J(L), M(L), \leq)$ is **reduced** and **separating**

How many reduced separating contexts are there with $|W| = m$ and $|W'| = n$?

$m = 0$, $n = 0$: $\mathbb{W}_1 = (\emptyset, \emptyset, \emptyset)$, $\mathbb{W}^+ =$ trivial lattice
$m = 0$, $n = 1$: no RS-contexts
$m = 1$, $n = 1$: $\mathbb{W}_2 = (\{0\}, \{0\}, \emptyset)$, $\mathbb{W}^+ =$ 2-element lattice
$m = 1$, $n = 2$: no RS-contexts

$m = 2$, $n = 2$: $\mathbb{W}_3 = (\{0,1\}, \{0,1\}, \{(0,0)\})$, $\mathbb{W}^+ =$ 3-elt lattice

$\mathbb{W}_4 = (\{0,1\}, \{0,1\}, \{(0,0), (1,1)\})$, $\mathbb{W}^+ =$ 4-element lattice

Counting finite reduced separating contexts

No reduced contexts for $m = 2$, $n = 3$

Number of reduced separating contexts (up to isomorphism)

	$n = 3$	4	5	6	7	8	9
$m = 3$	7	2	0	0	0	0	0
4	2	45	50	25	4	0	0
5	0	50	717	2241	3670	3598	2181
6	0	25	2241	37535	266178	?	?
7	0	4	3670	266178	?	?	?
8	0	0	3598	?	?	?	?
9	0	0	2181	?	?	?	?

(semi)distributive and selfdual lattices lie along the diagonal

Contexts for lattices of equivalence relations

Let $\underline{n} = \{0, 1, \ldots, n-1\}$, $S^{[2]}$ = two element subsets of S

Context of $Equ(n)$ is $(\underline{n}^{[2]}, \mathcal{P}(\underline{n-1}) - \{\emptyset\}, I)$

where $\{a, b\}\ I\ S$ iff $a, b \in S$ or $a, b \notin S$

Can use search over compatible relations to check if a given finite lattice is a sublattice of $Equ(n)$

Aim: find an efficient method for embedding a finite lattice into a (smallest) lattice of equivalence relations

Residuated frames

Recall: a **residuated lattice** is an algebra $\mathbb{A} = (A, \vee, \wedge, \cdot, \backslash, /, 1)$ such that $(A, \vee, \wedge)$ is a **lattice**, $(A, \cdot, 1)$ is a **monoid** and for all $x, y, z \in A \quad xy \le z \iff y \le x\backslash z \iff x \le z/y$.

A **residuated frame** is a 2-sorted ternary relational structure $\mathbb{W} = (W, W', I, \circ, \backslash\backslash, /\!/, E)$ such that (W, W', I) is a context, $\circ \subseteq W^3$, $\backslash\backslash \subseteq W \times W' \times W$, $/\!/ \subseteq W' \times W^2$ and for all $x, y, z \in W$ and $w \in W'$

1. $\gamma(E \circ x) = \gamma(\{x\}) = \gamma(x \circ E)$
2. $\gamma((x \circ y) \circ z) = \gamma(x \circ (y \circ z))$
3. $(x \circ y) I w \iff y I (x \backslash\backslash w) \iff x I (w /\!/ y)$ ($\circ$ is nuclear).

NB $x \circ y = \{z : \circ(x, y, z)\}$, $X \circ Y = \{z : \circ(x, y, z), x \in X, y \in Y\}$
Note that 1.-3. can be written as 1st-order formulas on $\mathbb{W}$

3. **corresponds** to $\circ_\gamma$ being a residuated operation on $\mathbb{W}^+$

Morphisms for residuated frames

Let $\mathbb{V}, \mathbb{W}$ be residuated frames. A relation $R \subseteq W \times W'$ is a morphism $R : \mathbb{V} \to \mathbb{W}$ if R is **lattice compatible** and

1. $R^{\downarrow} I^{\uparrow}(x \circ_{\mathbb{W}} y) = (R^{\downarrow} I^{\uparrow}\{x\}) \circ_{\mathbb{V}} (R^{\downarrow} I^{\uparrow}\{y\})$
2. $R^{\downarrow} I^{\uparrow}(x \backslash_{\mathbb{W}} I^{\downarrow} y) = (R^{\downarrow} I^{\uparrow}\{x\}) \backslash_{\mathbb{V}} (R^{\downarrow}\{y\})$
3. $R^{\downarrow} I^{\uparrow}(I^{\downarrow} x /_{\mathbb{W}} y) = (R^{\downarrow}\{x\}) /_{\mathbb{V}} (R^{\downarrow} I^{\uparrow}\{y\})$
4. $R^{\downarrow} I^{\uparrow}(E_{\mathbb{W}}) = \gamma(E_{\mathbb{V}})$

Theorem. The category of reduced separating residuated frames and morphisms is dually equivalent to the category of complete perfect residuated lattices and homomorphisms.

Use topological contexts to get a dual equivalence with all residuated lattices

Correspondence examples

Recall:

$\mathbb{A}$ is **integral** if it satisfies $x \leq 1$

$\mathbb{A}$ is **divisible** if it satisfies $x \leq y \implies x = y(y\backslash x) = (x/y)y$

$\mathbb{A}$ is **commutative** if it satisfies $xy = yx$

$\mathbb{A}$ is **modular** if it satisfies $x \leq z \implies (x \vee y) \wedge z = x \vee (y \wedge z)$

$\mathbb{A}$ is **distributive** if it satisfies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$\mathbb{A}$ is **bounded** if it has a constant $\bot$ that satisfies $\bot \leq x$

Let $\mathbb{W}$ be a residuated frame. Then $\mathbb{W}^+$ is **integral** iff

$\gamma(E_{\mathbb{W}}) = W$ iff $\forall x \in W, y \in W'(\forall z \in E_{\mathbb{W}}(zIy) \Rightarrow xIy)$

$\mathbb{W}^+$ is **commutative** iff $\forall x, y, z \in W(\circ(z, x, y) \iff \circ(z, y, x))$

For the correspondent of **distributivity**, expand the signature of a residuated frame with a ternary relation $R_\wedge$ for the $\wedge$ operator and its residuals $R_\backslash, R_/$, then state that $R_\wedge$ is associative, commutative, idempotent and nuclear over $\mathbb{W}$ (all first-order).

For finite contexts, there is a simpler first-order condition

Divisibility is not canonical [Gehrke, Priestley 2002], so has no first-order correspondent

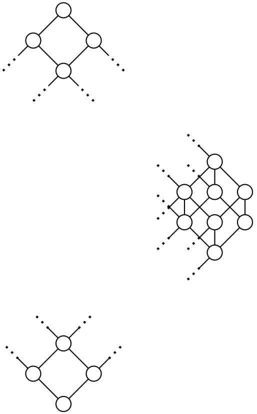
Consider the Chang algebra $\mathbb{C} = \{b_0 < b_1 < \cdots < a_1 < a_0\}$ with $a_i \cdot a_j = a_{i+j}$, $a_i \cdot b_j = b_j \cdot a_i = b_{\max(0,j-i)}$, $b_i \cdot b_j = b_0$

Then $\mathbb{C}$ is a divisible residuated lattice, but $\mathbb{C}^\delta = \{b_0 < b_1 < \cdots d < c \cdots < a_1 < a_0\}$ where $c \cdot c = c$, $c \cdot d = b_0$ so divisibility fails since $d < c$ but $d \neq c \cdot x$.

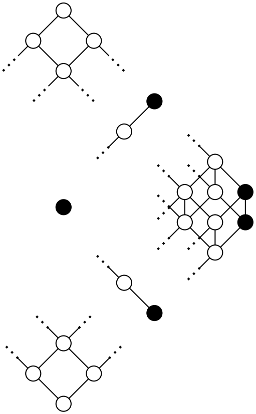
Modularity is not canonical

First look at how the MacNeille completion of a distributive lattice can fail to be modular [N. Funayama 1944]

Let L be the following sublattice of $\mathbb{C}^2 \times 2$ (ignore fusion)

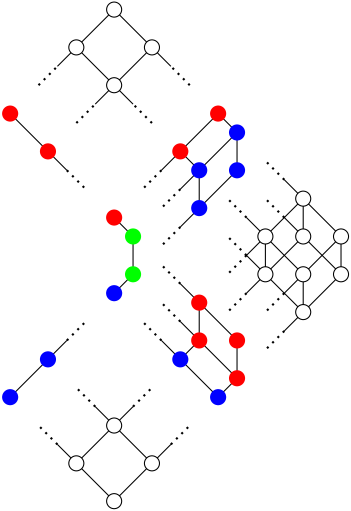


L



$\overline{L}$

The canonical extension of L



L^{δ}

Modularity is not canonical

Can we find a pentagon in M^δ for some modular lattice M?
No simple proof known; here is J. Harding's 1998 argument

Let M be the **modular** lattice of **finite and cofinite dimensional subspaces** of an infinite dimensional Hilbert space.

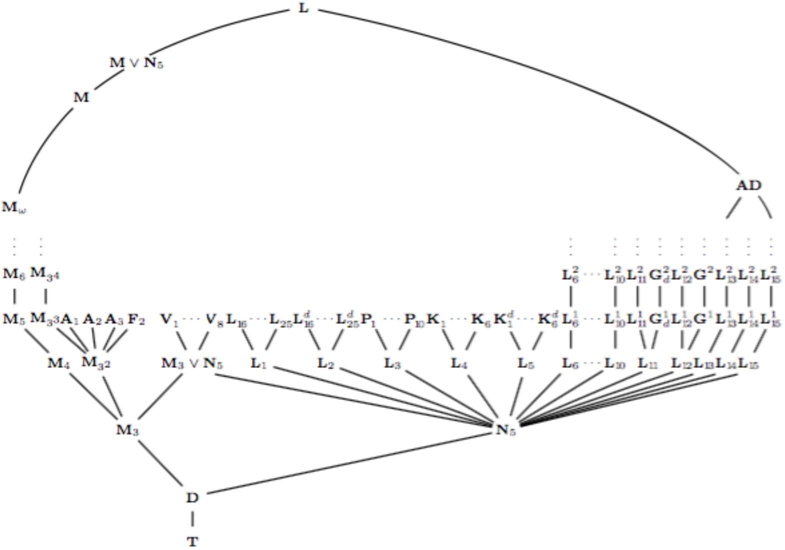
[Von Neumann 1936] A **continuous geometry** is a complete modular lattice L with a function $D : L \to [0, 1]$ that has finite range or is surjective and
if $a < b$ then $D(a) < D(b)$ $\quad D(a \vee b) + D(a \wedge b) = D(a) + D(b)$
$D(a) = 0$ if and only if $a = 0$, $\quad D(a) = 1$ if and only if $a = 1$,
and D agrees on elements with a common complement.

[Kaplansky 1955] Any orthocomplemented complete modular lattice is a **continuous geometry**

So **if** M^δ is modular, it must be a continuous geometry

But M^δ has infinitely many atoms, which leads to a contradiction

Other lattice varieties



Open Problems

Give an elementary proof that modularity is not canonical

Axiomatize the variety generated by canonical extensions of MV-algebras

Axiomatize the variety generated by canonical extensions of ℓ-groups

Is n-distributivity canonical?

Is join-semidistributivity canonical?

Is almost distributivity canonical? Does it have a 1st-order correspondent?

Does Var(N_5) have an equational basis that has a 1st-order correspondent?

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Thank You