Unary-determined distributive lattice-ordered magmas

Natanael Alpay, Peter Jipsen* and Melissa Sugimoto

Chapman University

NMSU Online Algebra Seminar 2021

October 18, 2021

Some brief history

Boolean Algebras with Operators: BAOs
Jónsson-Tarski 1951, 1952, Goldblatt 1989, Jónsson 1993

Distributive Lattices with Operators: DLOs
Goldblatt 1989, Gehrke-Jónsson 1994, 2000, 2004

Lattices with Operators: LOs
Gehrke-Harding 2001, Dunn-Gehrke-Palmigiano 2005

Heyting Algebras with Operators: HAOs
Bezhanishvili 1998, 1999 (monadic), Hasimoto 2001 (unary operators), Orlowska-Rewitzky 2007

Some brief history

Bjarni Jónsson was professor at Vanderbilt University 1966-1993

Mai Gehrke was a postdoc at Vanderbilt University 1988-1990
Then moved to New Mexico State University

John Harding was a postdoc at Vanderbilt University 1991-1993
Then moved to New Mexico State University

Guram Bezhanishvili was a PhD student of Leo Esakia and Hiroakira Ono until 1998, published three papers on monadic Heyting algebras by 2000
Then moved to New Mexico State University

I was a PhD student at Vanderbilt University 1987-1992
I should have moved to New Mexico State University

To each their own duality and logic

BAOs have a duality based on **Stone spaces**
Algebraic semantics of **polymodal logics**

DLOs have a duality based on **Priestley spaces**
Algebraic semantics of **positive polymodal logics**

LOs have a duality based on **topological polarities** (contexts)
Algebraic semantics of **nondistributive positive polymodal logics**

HLOs have a duality based on **Esakia spaces**
Algebraic semantics of **intuitionistic polymodal logics**

Some search metrics

Search query	MathSciNet	Google Scholar
Boolean algebras with operators	104	1990
Heyting algebras with operators	7	101
Distributive lattices with operators	12	329
Lattices with operators	14	112

Next we consider bounded distributive lattices and Heyting algebras

with a binary operator or one or two unary operators.

Distributive lattice-ordered magmas

Definition

A **distributive lattice-ordered magma** ($d\ell$-**magma** for short) $(A, \wedge, \vee, \bot, \top, \cdot)$ is a bounded distributive lattice with a binary operation $\cdot$ such that for all $x, y, z \in A$

$$x\cdot(y \vee z) = x\cdot y \vee x\cdot z \qquad x\cdot\bot = \bot$$
$$(x \vee y)\cdot z = x\cdot z \vee y\cdot z \qquad \bot\cdot x = \bot$$

A $d\ell$-**monoid** is a $d\ell$-magma with $1 \in A$ such that $(A, \cdot, 1)$ is a monoid.

A $d\ell$-magma is **commutative** if $x \cdot y = y \cdot x$.

$\wedge$-free reducts of $d\ell$-monoids are **(additively) idempotent semirings**.

Complete and completely join-preserving $d\ell$-monoids are **unital quantales** and they expand uniquely to **complete distributive residuated lattices**.

Finite distributive lattice-ordered magmas

Up to isomorphism, there are **many** finite $d\ell$-magmas:

2 of size 2

20 of size 3

1116 of size 4

Restricting to $d\ell$-monoids helps: let $f_n =$ number of algebras of size n

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 20,\ f_5 = 115,\ f_6 = 899,\ f_7 = 7782,\ f_8 = 80468$

A binary operation $\cdot$ is **idempotent** if $x \cdot x = x$.

Unary-determined $d\ell$-magmas

Definition

A $d\ell$-magma is **unary-determined** if $x{\cdot}y = (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$.

A **Boolean magma** is a $d\ell$-magma that has a complement operation $\neg$ s.t.

$$x \wedge \neg x = \bot \quad \text{and} \quad x \vee \neg x = \top.$$

Theorem

Every idempotent Boolean magma is unary-determined.

Proof.

$(x \wedge y){\cdot}(x \wedge y) \le x{\cdot}y \le (x \vee y){\cdot}(x \vee y)$ since $\cdot$ is order-preserving.

Therefore idempotence $\iff$ $x \wedge y \le x{\cdot}y \le x \vee y$.

Now $x{\cdot}\top \wedge y = x{\cdot}(y \vee \neg y) \wedge y = (x{\cdot}y \wedge y) \vee (x{\cdot}(\neg y)) \wedge y)$

$\le x{\cdot}y \vee ((x \vee \neg y) \wedge y) = x{\cdot}y \vee (x \wedge y) \vee (\neg y \wedge y) = x{\cdot}y$ □

Idempotent Boolean BI-algebras are unary-determined

Proof (continued).

Similarly $x \wedge \top y \leq x{\cdot}y$, hence $x{\cdot}y \geq (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$.

The opposite inequality $x{\cdot}y \leq (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$ is equivalent to

$$x{\cdot}y \wedge \neg(x{\cdot}\top \wedge y) \leq x \wedge \top{\cdot}y$$

$$\iff (x{\cdot}y \wedge \neg(x{\cdot}\top)) \vee (x{\cdot}y \wedge \neg y) \leq x \wedge \top{\cdot}y$$

$$\iff (x{\cdot}y \wedge \neg y) \leq x \wedge \top{\cdot}y \quad \text{since } x{\cdot}y \leq x{\cdot}\top$$

By idempotence, $x{\cdot}y \wedge \neg y \leq (x \vee y) \wedge \neg y = (x \wedge \neg y) \vee (y \wedge \neg y) \leq x$ and $x{\cdot}y \wedge \neg y \leq x{\cdot}y \leq \top{\cdot}y$. □

Term-equivalence for unary-determined $d\ell$-magmas

Definition

A $d\ell pq$-**algebra** $(A, \wedge, \vee, \bot, \top, p, q)$ is a bounded distributive lattice with two unary operations p, q that satisfy

$$p\bot = \bot \qquad p(x \vee y) = px \vee py \qquad x \wedge p\top \leq qx$$
$$q\bot = \bot \qquad q(x \vee y) = qx \vee qy \qquad x \wedge q\top \leq px$$

Unary-determined $d\ell$-magmas are term-equivalent to $d\ell pq$-algebras:

Theorem

1. *Let* **A** *be a $d\ell pq$-algebra and define $x{\cdot}y = (px \wedge y) \vee (x \wedge qy)$. Then $(A, \wedge, \vee, \bot, \top, \cdot)$ is a $d\ell$-magma that is unary-determined and p, q are definable as $px = x{\cdot}\top$ and $qx = \top{\cdot}x$.*
2. *Let* **A** *be a unary-determined $d\ell$-magma and define $px{=}x{\cdot}\top$, $qx{=}\top{\cdot}x$. Then $(A, \wedge, \vee, \bot, \top, p, q)$ is a $d\ell pq$-algebra and $\cdot$ is definable as $x{\cdot}y = (px \wedge y) \vee (x \wedge qy)$.*

Associativity, Commutativity, Idempotence from p, q

Theorem

Let $(A, \wedge, \vee, \bot, \top, p, q)$ be a $d\ell pq$-algebra and $x \cdot y = (px \wedge y) \vee (x \wedge qy)$.

1. *The operation $\cdot$ is commutative if and only if $p = q$.*
2. *If $p = q$ then $\cdot$ is associative if and only if*
   $$p((px \wedge y) \vee (x \wedge py)) = (px \wedge py) \vee (x \wedge ppy).$$
3. *If $p = q$ and $x \leq px = ppx$ then $\cdot$ is associative if and only if*
   $$px \wedge py \leq p((px \wedge y) \vee (x \wedge py)).$$
4. *The operation $\cdot$ is idempotent if and only if $x \leq px$ and $x \leq qx$, if and only if $p\top = \top = q\top$.*
5. *The operation $\cdot$ has an identity 1 if and only if $p1 = \top = q1$ and $(px \vee qx) \wedge 1 \leq x$.*
6. *If $\cdot$ has an identity then $\cdot$ is idempotent.*

Heyting algebras and bunched implication algebras

Definition

A **Heyting algebra** $(A, \wedge, \vee, \bot, \top, \to)$ is a bounded lattice $(A, \wedge, \vee, \bot, \top)$ such that $\to$ is the residual of $\wedge$, i. e.,

$$x \wedge y \leq z \iff y \leq x \to z.$$

The residual $\to$ ensures that the lattice is **distributive**.

Definition

A **bunched implication algebra** (BI-algebra) $(A, \wedge, \vee, \bot, \top, \to, *, 1, -\!*)$ is a Heyting algebra $(A, \wedge, \vee, \bot, \top, \to)$ such that $(A, *, 1)$ is a commutative monoid and $-\!*$ is the residual of $*$, i. e.,

$$x * y \leq z \iff y \leq x -\!* z.$$

The class of Heyting algebras and BI-algebras can both be defined by equations, so they are **varieties**.

BI-algebras and BI-logic

Bunched implication algebras are the algebraic semantics of **BI-logic**

BI-logic is the propositional part of **separation logic**, which is a Hoare logic for reasoning about data structures, memory allocation and concurrent programs.

The structure of BI-algebras is not well understood.

Defining $\neg x = x \to \bot$ and adding $\neg\neg x = x$ to BI-algebras gives the variety of Boolean BI-algebras, which contains the variety CRA of commutative relation algebras.

Finite BI-algebras "=" finite commutative distributive residuated lattices.

Every BI-algebra has a commutative $d\ell$-monoid as a reduct.

Every finite commutative $d\ell$-monoid expands uniquely to a BI-algebra.

Aim: find easy-to-describe subvarieties of BI-algebras

A BI-algebra is **idempotent** if $x{*}x = x$.

Recall that a **preorder** P is a binary relation that is reflexive and transitive

Theorem (Alpay, J. 2020)

Every finite idempotent **Boolean** *BI-algebra is determined by a preorder P on the set of atoms such that*

P is a preorder forest: xPy and xPz implies yPz or zPy, and

P has singleton roots: xPy and yPx and $\forall z(xPz \implies zPx)$ implies $x = y$

Preorder forests with singleton roots are counted by an Euler transform:

n	1	2	3	4	5	6	7	8	9	10	11
f_n	1	2	5	14	41	127	402	1306	4314	14465	49054
idem. BBI f_n	1	1	0	2	0	0	0	5	0	0	0

Term equivalence for idemp. unary-determined BI-algebras

An operation p^* is the **residual** of p if $px \le y \iff x \le p^*y$ holds.

Theorem

1. *Let* $\mathbf{A}$ *be a Heyting algebra with an operation* p*, residual* p^* *and constant* 1 *such that* $px \wedge py \le p((px \wedge y) \vee (x \vee py))$*,* $x \le px = ppx$*,* $p1 = \top$ *and* $px \wedge 1 \le x$*.*

   Define $x{*}y = (px \wedge y) \vee (x \wedge py)$ *and* $x{-\!*}y = (px \to y) \wedge p^*(x \wedge y)$*.*

   Then $(A, \wedge, \vee, \bot, \top, \to, *, -\!*, 1)$ *is an idempotent unary-determined BI-algebra.*

2. *Let* $(A, \wedge, \vee, \top, \bot, \to, *, -\!*, 1)$ *be an idempotent unary-determined BI-algebra, and define* $px = \top{*}x$ *and* $p^*x = \top{-\!*}x$*.*

   Then $(A, \wedge, \vee, \to, \top, \bot, p, p^*, 1)$ *is a Heyting algebra with an operation* p *that has* p^* *as residual and satisfies* $x \le px = ppx$*,* $px \wedge py \le p((px \wedge y) \vee (x \vee py))$*,* $p1 = \top$ *and* $px \wedge 1 \le x$*.*

Distributive lattices (= Heyting algebras) of cardinality ≤ 6

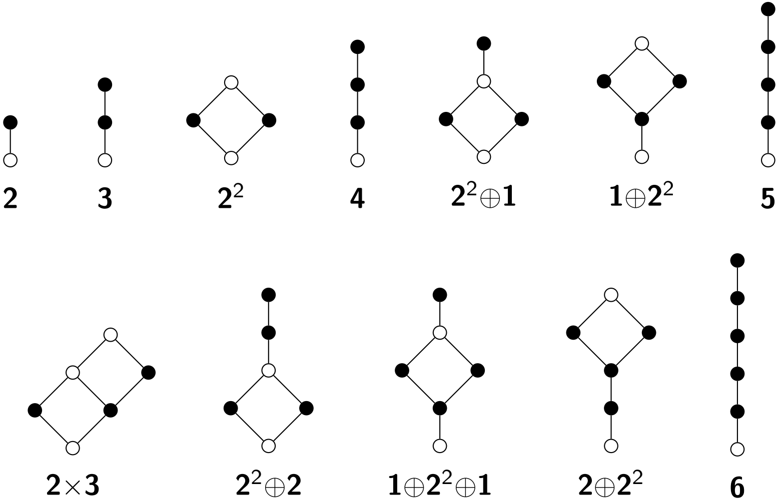

Figure: The completely join-irreducible elements are in black.

Downsets and completely join-irreducibles

Definition

Let $(W, \leq)$ be a poset. A **downset** is a subset $X \subseteq W$ such that $y \leq x \in X$ implies $y \in X$.

- Let $D(W, \leq)$ be the set of all downsets.
- The **downset lattice** is $(D(W, \leq), \cap, \cup, \emptyset, W)$.
- The downset lattice is a bounded distributive lattice.

Definition

An element x in a lattice A is **completely join-irreducible** if

$$x \neq \bigvee \{y \in A \mid y < x\}.$$

$J(A)$ denotes the set of completely join-irreducibles of A.

Kripke semantics for finite dℓ-magmas

Definition

$(J(A), \leq, R)$ is the **Birkhoff frame** of a finite $d\ell$-magma A with the ternary relation R defined by $R(x, y, z) \iff x \leq y{\cdot}z$.

- From $(x \vee y){\cdot}z = x{\cdot}z \vee y{\cdot}z$ it follows that $\cdot$ is order preserving.
- Hence R satisfies:

(R1) $u \leq x$ & $R(x, y, z) \implies R(u, y, z)$ (downward closure)
(R2) $R(x, y, z)$ & $y \leq v \implies R(x, v, z)$ (upward closure)
(R3) $R(x, y, z)$ & $z \leq w \implies R(x, y, w)$ (upward closure).

Definition

In general a **Birkhoff frame** $\mathbf{W} = (W, \leq, R)$ is a poset $(W, \leq)$ with a ternary relation $R \subseteq W^3$ that satisfies (R1), (R2), (R3).

The terminology "Birkhoff frame" is from [Galatos-J. 2017].

Birkhoff frames produce $d\ell$-magmas

Definition

For a Birkhoff frame $\mathbf{W}$ define the **downset algebra** $D(\mathbf{W}) = (D(W, \le), \cap, \cup, \cdot, \emptyset, W)$, where for $Y, Z \in D(W, \le)$

$$Y \cdot Z = \{x \in W \mid R(x, y, z) \text{ for some } y \in Y \text{ and } z \in Z\}.$$

$Y \cdot Z$ is a downset by (R1), (R2), (R3) of R.

Theorem

Let $\mathbf{W}$ *be a Birkhoff frame. Then*

- $D(\mathbf{W})$ *is a* $d\ell$*-magma.*
- $D(\mathbf{W})$ *is idempotent if and only if for all* $x, y, z \in W$, $R(x, x, x)$, *and* $(R(x, y, z) \implies x \le y \text{ or } x \le z)$.

PQ-Frames and P-Frames

Definition

$(W, \leq, P, Q)$ is a **PQ-frame** if

1. $(W, \leq)$ is a poset.
2. $u \leq x \ \& \ P(x, y) \ \& \ y \leq v \implies P(u, v)$
3. $u \leq x \ \& \ Q(x, y) \ \& \ y \leq v \implies Q(u, v)$

i.e., P, Q are **weakening relations**.

- A **P-frame** is a PQ-frame where $P = Q$.
- P is **reflexive** if $P(x, x)$ for all $x \in W$.
- P is **transitive** if $P(x, y) \ \& \ P(y, z) \implies P(x, z)$.

Note: $x \leq y \ \& \ P(y, y) \implies P(x, y)$ by weakening.

Correspondence theory for PQ-Frames and $d\ell pq$-algebras

Lemma

Let $\mathbf{W} = (W, \leq, P, Q)$ be a PQ-frame, and $\mathbf{A} = D(\mathbf{W})$ a $d\ell pq$-algebra. If it exists, the constant $1 \in A$ corresponds to a downset $E \subseteq W$. Then

1. *$x \leq px$ holds in $\mathbf{A}$ if and only if P is reflexive,*
2. *$ppx \leq px$ holds in $\mathbf{A}$ if and only if P is transitive,*
3. *$px = qx$ holds in $\mathbf{A}$ if and only if $P = Q$,*
4. *$p1 = \top$ holds in $\mathbf{A}$ if and only if $\forall x \exists y (y \in E \mathbin{\&} xPy)$ holds in $\mathbf{W}$,*
5. *$px \wedge 1 \leq x$ holds in $\mathbf{A}$ if and only if $x \in E \mathbin{\&} xPy \Rightarrow x \leq y$ in $\mathbf{W}$,*
6. *$px \wedge py \leq p((px \wedge y) \vee (x \wedge py))$ holds in $\mathbf{A}$ if and only if $wPx \mathbin{\&} wPy \Rightarrow \exists v (wPv \mathbin{\&} (vPx \mathbin{\&} v \leq y \text{ or } v \leq x \mathbin{\&} vPy))$ in $\mathbf{W}$.*

If $x \leq px = ppx$ then from (6) we get associativity in the term-equivalent $d\ell$-magma $(A, \wedge, \vee, \bot, \top, \cdot)$, where $x{\cdot}y = (px \wedge y) \vee (x \wedge py)$.

Preorder forest P-frames

A **preorder forest** P**-frame** is a P-frame such that P is a preorder (i. e. reflexive and transitive) and satisfies the formula

$$\text{(Pforest)} \quad xPy \text{ and } xPz \implies x \leq y \text{ or } x \leq z \text{ or } yPz \text{ or } zPy.$$

Theorem (main result; generalizes [Alpay, J. 2020])

Let $\mathbf{W} = (W, \leq, P)$ *be a preorder forest* P*-frame and* $D(\mathbf{W})$ *its corresponding downset algebra. Then*

- *the operation* $x{*}y = (px \wedge y) \vee (x \wedge py)$ *is associative in* $D(\mathbf{W})$,
- $E \subseteq W$ *is an identity element for* $*$ *in the downset algebra* $D(\mathbf{W})$ *if and only if* E *is a downset and* $pE = W$

  if and only if $(D(W, \leq), \cap, \cup, \emptyset, W, \to, *, \twoheadrightarrow, E)$ *is an idempotent unary-determined BI-algebra, where* $X \twoheadrightarrow Y = \{z \in W \mid X{*}\{z\} \subseteq Y\}$.

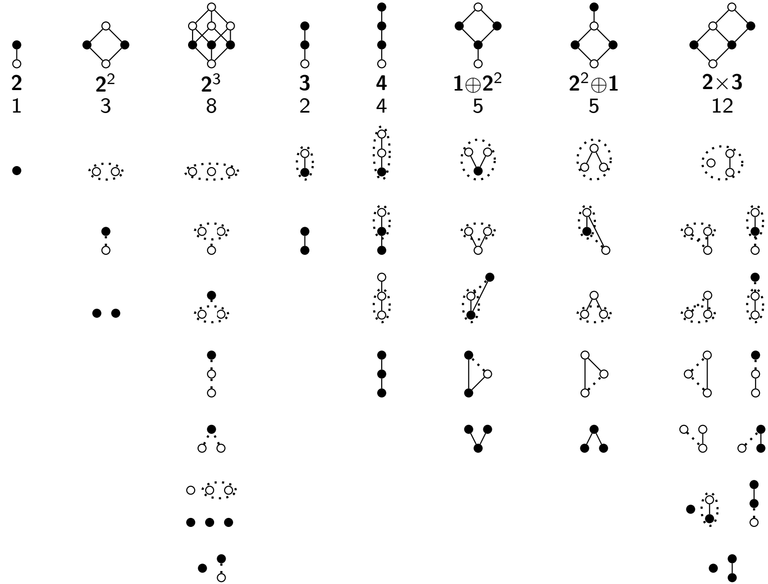

Figure: All 40 preorder forest P-frames $(W, \leq, P)$ with up to 3 join-irreducibles. Solid lines show $(W, \leq)$, dotted lines show the additional edges of P, and the identity (if it exists) is the set of black dots. The first row shows the lattice of downsets.

Conclusion

- Distributive lattices with unary operations are simpler than ones with binary operations. Hence the term-equivalence between unary-determined $d\ell$-magmas and $d\ell pq$-algebras is useful.
- We defined Birkhoff frames for $d\ell$-magmas, and PQ-frames for $d\ell pq$-algebras. These frames are logarithmic in size compared to the algebras.
- Preorder forest P-frames can be calculated more efficiently than idempotent unary-determined BI-algebras, and the P-frames can be drawn as Hasse diagrams of the poset (solid lines) and the preorder (dotted and solid lines).

n	1	2	3	4	5	6	7	8	9
all BI f_n	1	1	3	16	70	399	2261		
idem. BI f_n	1	1	2	6	15	44	115	326	
idem. u-d BI f_n	1	1	2	5	10	24	47	108	223

Open problems

Do commutative idempotent unary-determined $d\ell$-monoids have a decidable equational theory?

Do idempotent unary-determined BI-algebras have a decidable equational theory?

Find an axiomatization for the variety generated by residuated complex algebras of preorder forest P-frames.

How is this variety related to monadic Heyting algebras?

Bibliography

- N. Alpay, P. Jipsen: *Commutative doubly-idempotent semirings determined by chains and by preorder forests*, in proceedings of the 18th International Conference on Relational and Algebraic Methods in Computer Science (RAMiCS), LNCS Vol 12062, Springer (2020), 1–14.
- N. Alpay, P. Jipsen, M. Sugimoto: *Unary-determined distributive ℓ-magmas and bunched implication algebras*, in proceedings of the 19th International Conference on Relational and Algebraic Methods in Computer Science (RAMiCS), Springer (2021), to appear
- N. Galatos and P. Jipsen, *Distributive residuated frames and generalized bunched implication algebras*, Algebra Universalis, 78(3), (2017), 303–336
- N. Galatos and P. Jipsen, *The structure of generalized BI-algebras and weakening relation algebras*, Algebra Universalis, (2020) 81: 35
- W. McCune, Prover9 and Mace4, `http://www.cs.unm.edu/~mccune/Prover9`, 2005–2010.