Weakening relation algebras

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Outline

- Representable weakening relation algebras (RwkRA)
- Bounded cyclic distributive involutive residuated lattices (bCyDInRL)
- A game to characterize representations of weakening relation algebras
- Abstract weakening relation algebras

Representable **weakening** relation algebras

The **full algebra of binary relations on** X is

$$\mathbf{Rel}(X) = (\mathcal{P}(X^2), \cap, \cup, \emptyset, \top, ;, id_X, ^c, ^\smile) \text{ where } \top = X^2$$

$R; S$ = composition of R,S , $R^c = X^2 \setminus R$, and $R^\smile = \{(x, y) \mid (y, x) \in R\}$.

RRA = **representable relation algebras** = $\mathbb{SP}\{\mathbf{Rel}(X) \mid X \text{ is a set}\}$.

Tarski [1956] proved that **RRA** is a variety and Monk [1964] proved that **RRA** is not finitely axiomatizable.

For more details see the books by Givant [2017] and Maddux [2006].

The set of **weakening relations** on a poset $(X, \leq)$ is $Wk(X, \leq) = \{R \subseteq X^2 \mid \leq; R; \leq \, = R\}$.

The **full algebra of weakening relations on a poset** $(X, \leq)$ is $\mathbf{wk}(X, \leq) = (Wk(X, \leq), \cap, \cup, \emptyset, \top, ;, \leq, \sim)$ where $\sim R = X^2 \setminus R^\smile$.

The class of **representable weakening relation algebras** is **RwkRA** = $\mathbb{SP}\{\mathbf{wk}(X, \leq) \mid (X, \leq) \text{ is a poset}\}$.

A small example

Let $\mathbf{C}_2 = \{0, 1\}$ be the two element chain with $0 < 1$.

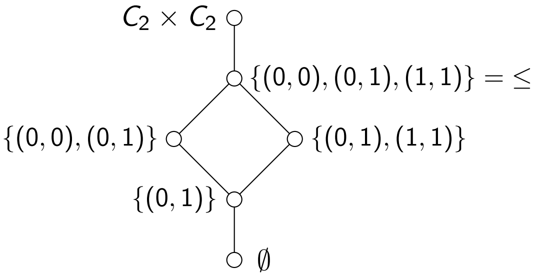


Figure: The full weakening relation algebra $\mathbf{wk}(\mathbf{C}_2)$

A slightly bigger example

For the 3-element chain $\mathbf{C}_3 = \{0, 1, 2\}$, $\mathbf{wk}(\mathbf{C}_3)$ has 20 elements

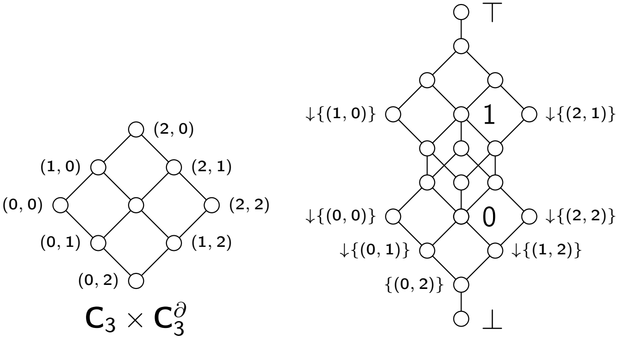
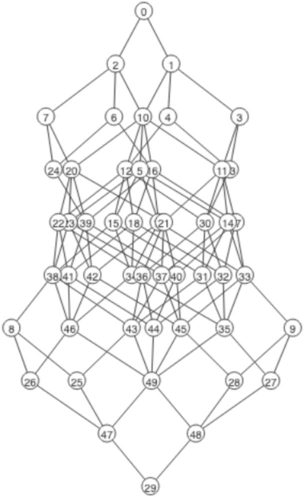


Figure: The full weakening relation algebra $\mathbf{wk}(\mathbf{C}_3)$

An even bigger example: **wk**(V)

For the 3-element poset that has the shape of a V, **wk**(V) has 50 elements.



Why study RwkRA?

Every **RRA** is a **RwkRA** with respect to the antichain order.

Symmetric relations ($R^{\smile} = R$) satisfy $\sim R = R^c$ so weakening RAs of symmetric relations are in RRA.

The operation $\sim$ of **complement-converse** is powerful enough to define residuals:

$$x\backslash y = \sim(\sim y; x) \text{ and } x/y = \sim(y; \sim x)$$

Algebras in **RwkRA** are **bounded distributive cyclic involutive residuated lattices**.

cyclic involutive means: $\sim x = x\backslash\sim 1 = \sim 1/x$ and $\sim\sim x = x$, where 1 is the identity element

Abstract relation algebras

Alfred Tarski [1941] gave a set of axioms, refined in 1943 to **10 equational axioms**, for (abstract) **relation algebras**

Jónsson-Tarski [1948 Bulletin of the AMS, Abstract 89]:

A **relation algebra** (**RA**) A is a **Boolean algebra** with a binary **associative operator** ; and a unary **operator** $^\smile$ such that ; has the **unit element** 1, $x^{\smile\smile} = x$, $(x; y)^\smile = y^\smile; x^\smile$ and $x^\smile; \neg(x; y) \leq \neg y$

Independence of the basis for RA

Joint work with Hajnal Andréka, Steve Givant, István Németi [JSL, 2017]

(R1) $x + y = y + x$
(R2) $x + (y + z) = (x + y) + z$
(R3) $\neg(\neg x + y) \vee \neg(\neg x + \neg y) = x$
(R4) $x;(y;z) = (x;y);z$
(R5) $x;1 = x$
(R6) $x^{\smile\smile} = x$
(R7) $(x;y)^{\smile} = y^{\smile};x^{\smile}$
(R8) $(x + y);z = x;z + y;z$
(R9) $(x + y)^{\smile} = x^{\smile} + y^{\smile}$
(R10) $x^{\smile};\neg(x;y) + \neg y = \neg y$

Theorem (Andréka, Givant, J., Németi 2017)

The identities (R1)-(R10) *are an independent basis for RA.*

For all i we find algebras A_i where (Ri) fails and the other identities hold.

Somewhat surprisingly, it turns out that by modifying (R8) slightly, (R7) becomes redundant. The following is also a basis:

(R1)-(R6),(R9),(R10) and (R8') $= x;(y + z) = x;y + x;z$

Bounded distributive cyclic involutive residuated lattices

$\mathbf{A} = (A, \cdot, +, \bot, \top, ;, 1, \sim)$ is in **bDCyInRL** if $(A, \cdot, +, \bot, \top)$ is a **bounded distributive lattice**, $(A, ;, 1)$ is a **monoid** and

$$x;y \le z \iff y;{\sim}z \le {\sim}x \iff {\sim}z;x \le {\sim}y$$

What other formulas do representable weakening relation algebras satisfy?

The $\{+, ;, \sim, 1\}$-subreducts of **RRA** are weakening relation algebras with respect to an antichain poset.
Since $1 = id_X$ we call them **diagonally representable** weakening RAs.

Lemma

The algebra $\mathbf{wk}(X, \le)$ *is diagonally representable if and only if* $1 \cap {\sim}1 = \bot$.

Proof.

$\Rightarrow$ Assume $\mathbf{wk}(X, \le)$ if diagonally representable. Then $\mathrm{id}_X \cap {\sim}\mathrm{id}_X = \bot$ since id_X is symmetric.
$\Leftarrow$ Assume $1 \cap {\sim}1 = \bot$ in $\mathbf{wk}(X, \le)$. Then $x \le y \implies (x, y) \notin {\sim}{\le}$, hence $x \ge y$, and therefore $x = y$. □

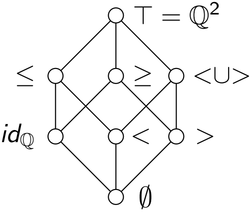
Examples of small weakening RAs

The **point algebra** is a relation algebra with 3 atoms $id_{\mathbb{Q}}, <, >$ where $<$ is the strict order on the rational numbers $\mathbb{Q}$.

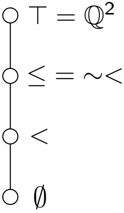
It has two **weakening subalgebras**: $\mathbf{S}_4 = \{\emptyset, <, \leq, \top\}$ and $\mathbf{A} = \{\emptyset, id_{\mathbb{Q}}, <, \leq, {<}{\cup}{>}, \top\}$.

Like the point algebra, both can only be represented on an **infinite** set.

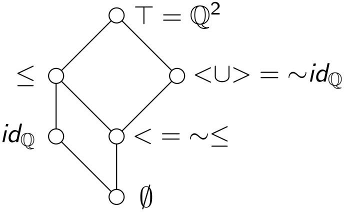
Note that **A** is **diagonally representable**, while $\mathbf{S}_4$ is not.



The point algebra



$\mathbf{S}_4$



A

Bounded even Sugihara monoids are in RwkRA

The variety **bSMon** of **bounded Sugihara monoids** is defined as **idempotent** $(x; x = x)$ **bDCyInRL**.

They are the algebraic semantics for the **relevance logic RM**$_t$.

$\mathbf{S}_4$ is a finite bounded Sugihara monoid with an **even** number of elements.

Lemma

Let **A** *be a finite linearly ordered bounded Sugihara monoid with an even number of elements. Then* **A** *is in* ***RwkRA****.*

Lemma

Algebras in ***RwkRA*** *satisfy* $x = {\sim}x \implies y = \top$, *hence every nontrivial algebra in RwkRA has an even number of elements.*

The 3-element Sugihara monoid $\mathbf{S}_3$ is a homomorphic image of $\mathbf{S}_4$, hence **RwkRA** is **not closed under homomorphic images**.

RwkRA is a quasivariety

The set $Wk(\mathbf{X})$ for a poset $\mathbf{X} = (X, \leq)$ is the set of upsets in $\mathbf{X}^\partial \times \mathbf{X}$.

$\implies$ $Wk(\mathbf{X})$ is complete and completely distributive, hence closed under Heyting implication.

Define $\mathbf{Wk}(X, \leq) = (Wk(X, \leq), \cap, \cup, \emptyset, \top, ;, \leq, \sim, \to)$ and $\mathbf{RWkRA} = \mathbb{SP}\{\mathbf{Wk}(\mathbf{X}) \mid \mathbf{X}$ is a poset$\}$.

Theorem

[Galatos-J. 2020] ***RWkRA*** *is a discriminator variety and* ***RRA*** *is the subvariety defined by* $\neg\neg x = x$ *where* $\neg x = x \to \emptyset$.

Now RwkRA is the class of subreducts of RWkRA (leaving out $\to$), hence RwkRA is a quasivariety.

Since it is not closed under $\mathbb{H}$, it is not a variety.

It also does not generate a discriminator variety since it contains subdirectly irreducible algebras that are not simple.

A game for representability by weakening relations

Let $\mathbf{A} = (A, +, \cdot, \bot, \top, \cdot, 1, \sim)$ be a bounded distributive cyclic involutive residuated lattice. When is $\mathbf{A}$ a member of **RwkRA**?

A **network** (N, λ) is a set N of nodes and a labelling function $\lambda : N^2 \to \mathcal{P}(A)$ such that $1 \in \lambda(x, x)$ and $\top \in \lambda(x, y)$ for all $x, y \in N$.

A network is **inconsistent** if for some $a \in A$, $x, y \in N$ we have $a \in \lambda(x, y)$ and $\sim a \in \lambda(y, x)$.

Networks are **partially ordered** by $(N, \lambda) \subseteq (N', \lambda') \iff N \subseteq N'$ and $\lambda(x, y) \subseteq \lambda'(x, y)$ for all $x, y \in N$.

An n-round **representation game** $\Gamma_n(\mathbf{A})$ is played by $\forall$ and $\exists$ over $n + 1$ moves.

After the ith move, $\exists$ returns a network $\mathbf{N}_i = (N_i, \lambda_i)$ such that $\mathbf{N}_0 \subseteq \mathbf{N}_1 \subseteq \cdots \subseteq \mathbf{N}_n$.

$\forall$ **wins** if $\exists$ returns an **inconsistent network**, otherwise $\exists$ wins.

The moves of the game

On the initialization move $\forall$ chooses $a \not\leq b \in A$ and $\exists$ must return $\mathbf{N}_0$ with some $x, y \in N_0$ such that $a \in \lambda(x, y)$ and $\sim b \in \lambda(y, x)$.

On the ith move for $0 < i \leq n$, $\forall$ may challenge $\exists$ with any of the following moves.

- join move: $\forall$ picks $x, y \in N_{i-1}$, $a \in \lambda_{i-1}(x, y)$ and $b, c \in A$ such that $a \leq b + c$. $\exists$ must return $\mathbf{N}_i$ with $b \in \lambda_i(x, y)$ or $c \in \lambda_i(x, y)$.
- involution move: $\forall$ picks $x, y \in N_{i-1}$ and $a \in A$. $\exists$ must return $\mathbf{N}_i$ with $a \in \lambda_i(x, y)$ or $\sim a \in \lambda_i(y, x)$.
- composition move: $\forall$ picks $x, y, z \in N_{i-1}$, $a \in \lambda_{i-1}(x, y)$ and $b \in \lambda_{i-1}(y, z)$. $\exists$ must return $\mathbf{N}_i$ with $a; b \in \lambda_i(x, z)$.
- witness move: $\forall$ picks $x, y \in N_{i-1}$, $a \in \lambda_{i-1}(x, y)$ and $b, c \in A$ such that $a = b; c$. $\exists$ must return $\mathbf{N}_i$ with some $z \in N_i$ such that $b \in \lambda_i(x, z)$ and $c \in \lambda_i(z, y)$.

The game characterizes representability

Theorem

A *is in RwkRA if and only if* $\exists$ *has a winning strategy for* $\Gamma_\omega(\mathbf{A})$.

Here $\Gamma_\omega(\mathbf{A})$ denotes a game with countably many moves and $\exists$ wins if she never returns an inconsistent network.

In the forward direction, $\exists$ can use a representation $h : \mathbf{A} \to \mathbf{wk}(X, \leq)$ to define a network $\mathbf{N} = (X, \lambda)$ where $\lambda(x, y) = \{c \in A \mid (x, y) \in h(c)\}$, and then play this network at each turn.

For the reverse direction, a winning strategy for a countable **A** can be used to build a representation for **A**. This generalizes to uncountable structures by the downward Löwenheim Skolem Theorem.

What we learned from the game

We calculated all 64 algebras in **bDCyInRL** with up to **6 elements** and used the game to check if they are **representable** by weakening relations.

Lemma

Algebras in RwkRA satisfy the following quasiequations:

- $1 \leq x + {\sim}1 \implies y \leq x;y$
- $x \leq y;z \implies x \leq y;zw + (x;{\sim}w)y;z$
- $x \leq y;z \implies x \leq wy;z + y;({\sim}w;x)z$

We add these as axioms to **bDCyInRL** to define a quasivariety of **abstract weakening RAs**.

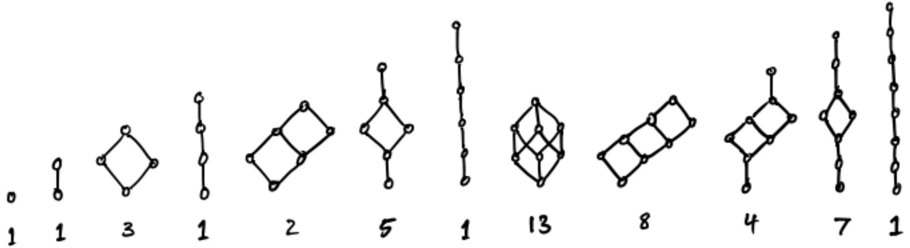
It follows that $(x;\top)1;x = x;1(\top;x) = x$ and $\top;x;\top;x;\top = \top;x;\top$.

Finding representations of all 8-element (wk)RAs

There are (up to isomorphism) 18 relation algebras with up to 8 elements.

They are all representable, and H. Andréka and R. Maddux [1994] determined all cardinalities of sets on which they are representable.

A current project is to do the same for the 47 (abstract) weakening RAs with up to 8 elements.



Other generalizations of RAs

István Németi [1987] proved that if one **removes or weakens associativity** of RAs, the resulting varieties **NA** of **nonassociative** RAs and **WA** of **weakly associative** RAs have **decidable** equational theories.

N. Galatos and J. [2013] showed that if **classical negation** in **RA** is weakened to a **De Morgan negation** then the resulting variety qRA of **quasi relation algebras** has a **decidable** equational theory.

However there are **no natural models using binary relations and composition** (other than the classical models in **RRA**).

RwkRA is a large class of natural models using binary relations and composition, while generalizing RRA.

Many techniques from the theory of relation algebras can be adapted to RwkRA or to the larger finitely based class of abstract weakening RAs.

This is your invitation to join in the exploration and have a great time!

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Thank you very much!