

Representability of small relation algebras and involutive FL-algebras

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Introduction

- Part 1 on representability of RA is joint work with [P. Hertz](#), [R. Kramer](#) and [R. Maddux](#)
- Aim: Give a survey of relation algebras with ≤ 16 elements, indicating why each is representable or not

- Part 2 on involutive residuated lattices is joint work with [N. Galatos](#)
- Aim: Clarify the connection with relation algebras and generalize the notion of representability

Tarski's variety of relation algebras

RA is the variety of algebras $(A, +, 0, \cdot, 1, ^-, ;, e, \smile)$ such that

- $(A, +, 0, \cdot, 1, ^-)$ is a **Boolean algebra**
- $(A, ;, e)$ is a **monoid**
- **converse** \smile is an **involution**: $x^{\smile\smile} = x$ and $(x;y)^{\smile} = y^{\smile};x^{\smile}$
- $;$ and \smile **distribute** over $+$
- $x^{\smile};(x;y)^- \leq y^-$

The last three are equivalent to:

$$(x;y) \cdot z = 0 \Leftrightarrow (x^{\smile};z) \cdot y = 0 \Leftrightarrow (z;y^{\smile}) \cdot x = 0$$

Concrete relation algebras

The variety RRA of *representable relation algebras* is generated by the *square relation algebras*

$$\text{Re}(U) = (\mathcal{P}(U^2), \cup, \emptyset, \cap, U^2, -, \circ, id_U, ^{-1})$$

where U is any set (closing under SP suffices)

Monk [1964] proved RRA is a **nonfinitely axiomatizable subvariety** of RA

Jónsson [1991] proved RRA cannot be axiomatized with **finitely** many variables

Hirsch and Hodkinson [2001] proved that it is **undecidable** whether a finite relation algebra is in RRA

Yet this decision has been made for many specific finite relation algebras

Group relation algebras

If $(G, \circ, e, {}^{-1})$ is a group, then the *complex algebra* of G is

$$\text{Cm}G = (\mathcal{P}(G), \cup, \emptyset, \cap, G, {}^{-1}, \circ, \{e\}, {}^{-1})$$

is a *group relation algebra* with $X \circ Y = \{x \circ y : x \in X, y \in Y\}$ and $X^{-1} = \{x^{-1} : x \in X\}$.

This algebra is in RRA since it is embedded in $\text{Re}(G)$ via the Cayley map $g \mapsto \{(x, xg) : x \in G\}$ (extended by distributivity to subsets of G).

McKenzie [1970] proved that the variety GRA, generated by all complex algebras of groups, is nonfinitely axiomatizable relative to RRA.

A relation algebra is *integral* if e is an atom.

Of the 115 integral RAs of size ≤ 16 , most representable ones are in GRA.

Group representations of small RAs

Maddux [2006] lists all integral RAs with ≤ 16 elements

Here are group representations for integral RAs of size ≤ 8

$$\begin{aligned} 1_1 &\cong \text{Cm}\mathbb{Z}_1 & 1_2 &\cong \text{Cm}\mathbb{Z}_2 & 2_2 &\cong \text{Sg}^{\text{Cm}\mathbb{Z}_3}(\{0\}) \\ 1_3 &\cong \text{Sg}^{\text{Cm}\mathbb{Q}}(\{r : r > 0\}) & 2_3 &\cong \text{Cm}\mathbb{Z}_3 & 3_3 &\cong \text{Sg}^{\text{Cm}\mathbb{Z}_7}(\{1, 2, 4\}) \\ 1_7 &\cong \text{Sg}^{\text{Cm}\mathbb{Z}_4}(\{2\}) & 2_7 &\cong \text{Sg}^{\text{Cm}\mathbb{Z}_6}(\{2, 4\}) & 3_7 &\cong \text{Sg}^{\text{Cm}\mathbb{Z}_6}(\{3\}) \\ 4_7 &\cong \text{Sg}^{\text{Cm}\mathbb{Z}_9}(\{3, 6\}) & 5_7 &\cong \text{Sg}^{\text{Cm}\mathbb{Z}_5}(\{1, 4\}) & 6_7 &\cong \text{Sg}^{\text{Cm}\mathbb{Z}_8}(\{1, 4, 7\}) \\ & & 7_7 &\cong \text{Sg}^{\text{Cm}\mathbb{Z}_3^2}(\{1, 2\} \times \{1, 2\}) \end{aligned}$$

E.g. 3_7

\circ	e	a	b
e	e	a	b
a	a	e	b
b	b	b	1

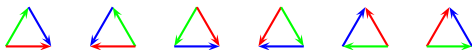
$$\begin{aligned} G &= \mathbb{Z}_6, & 1 &= e + a + b \\ e &= \{0\} = e^{-1} \\ a &= \{3\} = a^{-1} \\ b &= \{1, 2, 4, 5\} = b^{-1} \end{aligned}$$

Much information in the table is redundant

Consider triples (x, y, z) of atoms that satisfy $x; y \geq z$.

This condition is equivalent to $y^{\smile}; x^{\smile} \geq z^{\smile}$, $x^{\smile}; z \geq y$, $z^{\smile}; x \geq y^{\smile}$, $y; z^{\smile} \geq x^{\smile}$, $z; y^{\smile} \geq x$, hence we group these triples into a *cycle*

A cycle is represented compactly by a triangle of colored arrows (the converse elements are given by the reverse arrows)



For symmetric atoms a line is used instead of an arrow


1_3	r	r^{\smile}
r	r	1
r^{\smile}	1	r



3_7	a	b
a	e	b
b	b	1



An integral relation algebra with identity atom e contains an

identity cycle  if and only if $x = y$.

Hence algebras with the same number of atoms do not differ with respect to these cycles.

It follows that an algebra with a symmetric atom a (green) and two nonsymmetric atoms r, r^\smile (red arrow and reverse arrow) is determined by a subset of the following cycles:



Up to isomorphism there are 37 such integral RAs, numbered 1_{37} – 37_{37}

Listed by Maddux [2006] and Comer [1986] (different numbering)

Hypergraphs of relation algebras

Since e behaves the same for all integral relation algebras, this atom is omitted from the table, cycle list, and hypergraph.

The other atoms of the algebra are given by vertices of a directed hypergraph.

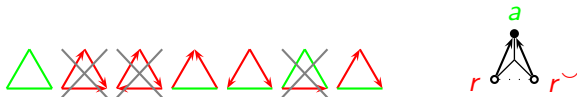
An **arrow points** from vertex a to b if $a; a^\smile \geq b$

A vertex a is colored **black** if $a; a \geq a^\smile$.

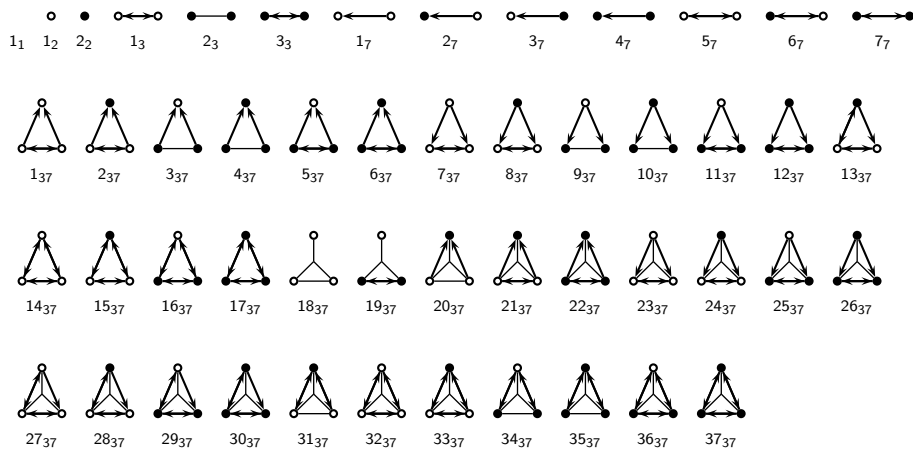
A **3-hyperedge** (thin lines) connects 3 vertices a, b, c if $a; b^\smile \geq c$.

Two atoms connected by a dotted line represent a converse pair r, r^\smile .

E.g. 20_{37} :



The first 50 integral relation algebras



The 65 symmetric integral relation algebras of size 16



Which of these algebras are in RRA or GRA?

The main methods for proving an algebra is representable are

- to construct the algebra from two smaller representable algebras by a *2-cycle extension* $\mathbf{A}[\mathbf{B}]$ (Comer [1983]) and
- a so-called *one-point extension* method, where it is shown that a (possibly infinite) representation can be built by adding one element at a time.

For integral algebras, the 2-cycle extension $\mathbf{A}[\mathbf{B}]$ has cycles defined by

- taking the union of the cycles in \mathbf{A} and \mathbf{B} and
- for all nonidentity atoms a in \mathbf{A} and b in \mathbf{B} add the cycle $b; b^\smile \geq a$

Graphically: add arrows from all vertices of \mathbf{B} to all vertices of \mathbf{A} .

Representations from 2-cycle products

Theorem (Comer (1983))

$\mathbf{A}, \mathbf{B} \in \text{RRA}$ (or GRA) if and only if $\mathbf{A}[\mathbf{B}] \in \text{RRA}$ (or GRA)

Using the earlier group representations for $1_1, 1_2, 2_2, 1_3-3_3, 5_7-7_7$, and 2-cycle extensions one can construct representations for the algebras $1_7-4_7, 1_{37}-12_{37}, 1_{65}-20_{65}$.

A *bidirectional 2-cycle extension* $\mathbf{A} \star \mathbf{B}$ has cycles defined by

- taking the union of the cycles in \mathbf{A} and \mathbf{B} and
- for all nonidentity atoms $x, y \in \mathbf{A} \cup \mathbf{B}$ add the cycle $x; x^{\sim} \geq y$

Graphically: add arrows between all vertices of \mathbf{A} and \mathbf{B} and make all dots black.

It is used to show that $15_{37}, 17_{37}, 24_{65}$ are representable.

Representations as edge-colored digraphs

A representation of an integral relation algebra is an embedding into $\text{Re}(X)$ for some X .

Equivalently, if the algebra has atoms r_0, r_1, \dots , a representation is a complete edge-colored digraph (X, R_0, R_1, \dots) with vertices $i, j \in X$ labeled by r_m if $(i, j) \in R_m$ such that

- excluded cycles do not appear in the graph, and
- for all vertices i, j , if the edge (i, j) is labeled by z and $x; y \geq z$ then there exists a vertex k such that (i, k) is labeled x and (k, j) is labeled y ,
i.e. each cycle must appear on each matching edge in the graph.

E.g.



represents 5_7 :



Representations by one-point extensions

A relation algebra has a *flexible atom* a if $a \leq x;y$ for all $x, y \notin \{0, e\}$

In the hypergraph a (and a^\smile) have all possible edges and hyperedges entering and leaving.

The one-point extension method applies to all algebras that have a flexible atom, hence 31_{37} , 33_{37} , 35_{37} , 36_{37} , 37_{37} , 32_{65} , 33_{65} , 34_{65} , 55_{65} , 57_{65} , 59_{65} , 61_{65} , 63_{65} , 64_{65} , 65_{65} are representable.

It also applies to several algebras that do not have a flexible atom, including 13_{37} , 23_{37} , 30_{37} , 30_{65} , 31_{65} , 52_{65} .

Additional finite group representations

$$18_{37} \cong \text{Cm}\mathbb{Z}_4$$

$$19_{37} \cong \text{Sg}^{\text{Cm}Q_8}(\{i, j, k\})$$

$$20_{37} \cong \text{Sg}^{\text{Cm}\mathbb{Z}_{12}}(\{1, 7, 9\})$$

$$22_{37} \cong \text{Sg}^{\text{Cm}(\mathbb{Z}_{16})}(\{1, 3, 5, 6, 7, 14\})$$

$$25_{65} \cong \text{Cm}\mathbb{Z}_2^2$$

$$26_{65} \cong \text{Sg}^{\text{Cm}\mathbb{Z}_6}(\{1, 5\})$$

$$27_{65} \cong \text{Sg}^{\text{Cm}\mathbb{Z}_{10}}(\{1, 2, 8, 9\})$$

$$28_{65} \cong \text{Sg}^{\text{Cm}D_{12}}(\{b, ab, a^3b\}) \text{ where } D_{12} = \langle a, b \mid a^6 = b^2 = e, ba = a^5b \rangle$$

$$29_{65} \cong \text{Sg}^{\text{Cm}(\mathbb{Z}_3 \times \mathbb{Z}_3)}(\{(0, 1), (0, 2)\}, \{(1, 0), (2, 0)\})$$

Further representations

$$39_{65} \cong \text{Sg}^{\text{Cm}\mathbb{Z}_7}(\{1, 6\})$$

$$46_{65} \cong \text{Sg}^{\text{Cm}\mathbb{Z}_{20}}(\{5, 6, 14, 15\})$$

$$53_{65} \cong \text{Sg}^{\text{Cm}\mathbb{Z}_2^4}(\{0001, 0010, 0011\}, \{0101, 0110, 1010, 1011, 1110, 1111\})$$

$$62_{65} \cong \text{Sg}^{\text{Cm}\mathbb{Z}_{13}}(\{1, 5, 8, 12\}, \{2, 3, 10, 11\})$$

Additional ad hoc infinite representations

51₆₅ by Comer [1986]

56₆₅ by Lukacs [1991]

These are all 71 representable integral relation algebras of 102 with size 16

Checking if a finite relation algebra is not representable

Theorem (Lyndon 1950, Maddux 1983)

*There is an algorithm that halts if a given finite relation algebra is **not** representable*

Lyndon gives a recursive axiomatization for RRA

Maddux defines a sequence of varieties RA_n such that $RA = RA_4 \supset RA_5 \supset \dots$ $RRA = \bigcap_{n \geq 4} RA_n$ and it is decidable if a finite algebra is in RA_n

Implemented as a GAP program [Jipsen 1993]

Relation algebras and logical games

Recall that a representation of an integral RA is a complete edge-coloured digraph such that each cycle from the algebra appears on each matching edge

[Hirsch Hodkinson 2002] express the construction of a representation as a two-player game, and the algebra is representable if the existential player has a winning strategy.

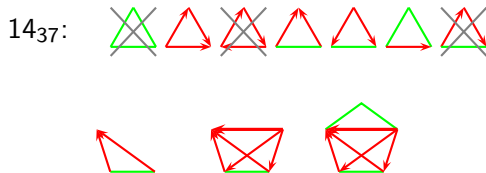
Given a partial representation, the universal player chooses an edge (i, j) with label z and two atoms x, y such that $x;y \geq z$ in the algebra.

The existential player adds a point k to the graph, labels (i, k) with x , (k, j) with y , and now has to find edge-labels for all the remaining edges (k, m) such that no excluded cycle is forced to appear in the partial representation.

A “ n -pebble” version of this game characterizes membership in RA_n

Example of a nonrepresentation game

E.g. [McKenzie 1970] showed that the 4-atom algebra 14_{37} is not representable



The resulting configuration is said to be *forbidden* from any representation, since the existential player cannot label all the edges using only the cycles in 14_{37}

By construction this particular configuration is also *required* in any representation of this algebra, a contradiction. Hence no representation exists.

Nonrepresentability results



Similar games on 5 points show that 16_{37} , 21_{37} , 24_{37} – 29_{37} , 34_{37} , 21_{65} , 22_{65} , 23_{65} , 35_{65} – 38_{65} , 40_{65} – 45_{65} , 47_{65} – 50_{65} , 54_{65} , 58_{65} are not in RA_5

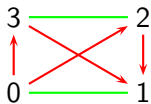
The last two algebras, 32_{37} and 60_{65} , were found to be in $RA_5 \setminus RA_6$

Theorem

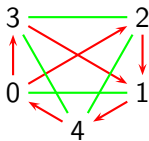
32_{37} is not representable.





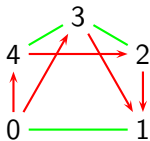
We show that there is no complete edge-labeled graph that contains all the required cycles on each matching edge and omits the green cycle  and the red cycle 



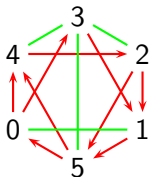
$C_1(0, 1, 2, 3)$





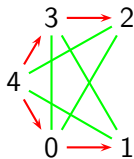
Claim 1: $C_1(0, 1, 2, 3)$ is forbidden. Else  should occur on edge $(0, 1)$. But then  occurs at $2, 3, 4$.



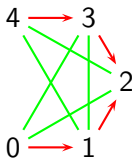
$C_2(0, 1, 2, 3, 4)$




Claim 2: $C_2(u, v, x, y, z)$ is forbidden. Else  should occur on edge $(0, 1)$. But then $3-4$ causes  to occur at $2, 3, 5$.



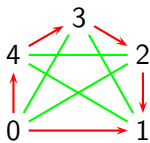
$C_3(0, 1, 2, 3, 4)$



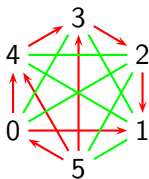
$C'_3(0, 1, 2, 3, 4)$



Claim 3: $C_3(0, 1, 2, 3, 4)$ and $C'_3(0, 1, 2, 3, 4)$ are forbidden. Suppose $C_3(0, 1, 2, 3, 4)$ occurs. If $2 \rightarrow 1$ then $C_2(4, 1, 2, 0, 3)$ occurs and if $1 \rightarrow 2$ then $C_2(4, 2, 1, 3, 0)$ occurs, both impossible by Claim 2. However $1 \text{---} 2$ is also impossible since 

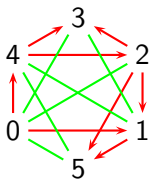
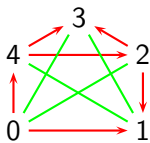
The proof for $C'_3(0, 1, 2, 3, 4)$ is analogous.





$C_4(0, 1, 2, 3, 4)$

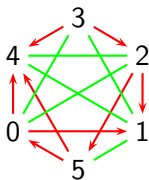
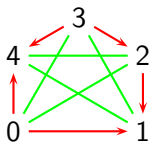


Claim 4: $C_4(0, 1, 2, 3, 4)$ is forbidden. Else  should occur on edge $(0, 1)$. Then $5 \rightarrow 4$ and $5 \rightarrow 3$. Now $2 \rightarrow 5$ implies  on $5, 3, 2$, while $5 \rightarrow 2$ implies $C_1(5, 1, 2, 0)$, and finally $2 \rightarrow 5$ implies $C_2(5, 1, 2, 0, 3)$.



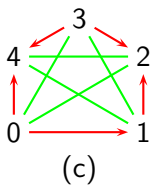
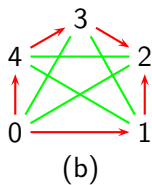
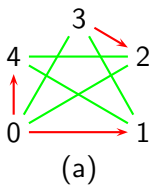
$C_5(0, 1, 2, 3, 4)$

Claim 5: $C_5(0, 1, 2, 3, 4)$ is forbidden. Else  should occur on edge $(0, 1)$, hence $2 \rightarrow 5$. Now $5 \rightarrow 4$ implies  on $5, 4, 2$, while $4 \rightarrow 5$ implies $C_1(0, 5, 1, 4)$, and finally $4 \rightarrow 5$ implies $C_3(5, 1, 2, 0, 3)$.



$C_6(0, 1, 2, 3, 4)$

Claim 6: $C_6(0, 1, 2, 3, 4)$ is forbidden. Else \triangle should occur on edge $(0, 1)$, hence $5 \rightarrow 4$ and $5 \rightarrow 3$. Now $5 \rightarrow 2$ implies $C_1(5, 1, 2, 0)$, while $2 \rightarrow 5$ implies $C'_3(5, 1, 3, 2, 0)$, and finally $2 \rightarrow 5$ implies $C_5(2, 1, 5, 4, 0)$.



Note that (a) is a *required* configuration. The missing edges must be red arrows but, depending on how they are oriented, we obtain one of the forbidden configurations C_3 , C_4 or C_6 .

This completes the proof that 32_{37} is nonrepresentable.

A similar but longer argument proves the same for 60_{65} .

Part 2: RAs as expansions of involutive FL-algebras

Joint work with N. Galatos

A *FL-algebra* is a residuated lattice $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ expanded with a constant 0 that can denote any element

The linear negations are defined as $\sim x = x \backslash 0$ and $-x = 0 / x$

A FL-algebra is *involutive* if $\sim -x = x = -\sim x$ and *cyclic* if $\sim x = -x$

We repeat the definition of relation algebras in a signature **close to residuated lattices** as algebras of the form $\mathbf{A} = (A, \wedge, \vee, ', \perp, \top, \cdot, 1, \smile)$, such that $(A, \wedge, \vee, ', \perp, \top)$ is a Boolean algebra $(A, \cdot, 1)$ is a monoid and for all $a, b, c \in A$

- 1 $(a^\smile)^\smile = a, \quad (ab)^\smile = b^\smile a^\smile$
- 2 $a(b \vee c) = ab \vee ac, \quad (a \vee b)^\smile = a^\smile \vee b^\smile$ and
- 3 $a^\smile(ab)' \leq b'$

FL'-algebras

Given a relation algebra \mathbf{A} , we define $a \setminus b = (a \smile b')'$, $b/a = (b' a \smile)'$ and $0 = 1'$.

Then $(\mathbf{A}, \wedge, \vee, \cdot, \setminus, /, 1, 0)$ is a cyclic involutive FL-algebra with $\sim x = -x = x' \smile = x \smile'$.

A *FL'-algebra* is an expansion of a FL-algebra with a unary operation $'$ that satisfies the conditions

- $x'' = x$
- $(x \vee y)' = x' \wedge y'$
- $\sim(x') = (\sim x)'$
- $-(x') = (-x)'$

Lemma

Every involutive FL'-algebra that satisfies $(xy)' = x' + y'$ is cyclic.

Proof.

Note that $x \setminus y = (\sim x + y)'' = (\sim x' \cdot y')'$, so

$1' = 1 \setminus 1' = (\sim 1' \cdot 1'')' = \sim 1 = 0$. (Similarly $x / y = (x' \cdot -y')'$.)

Moreover, for every x, y , we have $\sim x \leq y$ iff $y' \leq \sim x'$ iff $x'y' \leq 1' = 0$ iff $(x + y)' \leq 0$ iff $1 \leq x + y$ iff $-x \leq w$. Therefore $\sim x = -x$, for all x . \square

We also define two constants $\perp = 1 \wedge 1'$ and $\top = 1 \vee 1'$.

An involutive FL'-algebra is called *Boolean* if the reduct $(A, \wedge, \vee, ', \perp, \top)$ is a Boolean algebra, or equivalently if it is distributive and satisfies $x \vee x' = \top$.

Characterizing relation algebras

Lemma

An involutive FL'-algebra is (term equivalent to) a relation algebra iff it is Boolean and satisfies $(xy)' = x' + y'$.

Proof.

It is easy to check that every relation algebra satisfies the above properties. Conversely, assume that an involutive FL'-algebra satisfies the properties.

By the preceding lemma the algebra is cyclic. Define $x^U = \sim(x')$. We have $(x \vee y)^U = \sim(x \vee y)' = \sim(x' \wedge y') = (\sim x' \wedge \sim y') = x^U \vee y^U$.

Using the commutation of \sim and $'$, and cyclicity, we get

$$x^{UU} = \sim((\sim x)'') = \sim\sim x = \sim -x = x.$$

$$(xy)^U = \sim(xy)' = \sim(x' + y') = \sim y' \cdot \sim x' = y^U x^U.$$

To verify $x^U(xy)' \leq y'$ we rewrite it by applying converse on both sides to get the equivalent equation $(xy)^U x \leq y^U$, namely $-(xy)x \leq -y$ or $(0/(xy))x \leq 0/y$. This is equivalent to $0/(xy) \leq 0/(xy)$, hence true. \square

Skew relation algebras

A *skew relation algebra* is defined to be an involutive FL'-algebra that is Boolean.

Theorem

Skew relation algebras are term equivalent to algebras of the form
 $\mathbf{A} = (A, \wedge, \vee, ', \perp, \top, \cdot, 1, \cup, \sqcup)$, *such that* $(A, \wedge, \vee, ', \perp, \top)$ *is a Boolean algebra, $(A, \cdot, 1)$ is a monoid and for all* $a, b, c \in A$

- $(a^\cup)^\sqcup = a = (a^\sqcup)^\cup$ and
- $(a \cdot b) \wedge c = \perp \iff (b \cdot c^\cup)^\sqcup \wedge a = \perp \iff (c^\sqcup \cdot a)^\cup \wedge b = \perp$

The term equivalence is via $x^\cup = \sim x'$ and $x^\sqcup = -x'$

Representable π relation algebras

Given a set X and a bijection π on X we define the algebra

$\text{Re}(X, \pi) = (\mathcal{P}(X^2), \cup, \cap, \circ, \cup, \sqcup, id_X)$, where

\circ is relational composition,

$$R^\cup = \{(y, \pi(x)) : (x, y) \in R\} \text{ and}$$

$$R^\sqcup = \{(\pi^{-1}(y), x) : (x, y) \in R\}$$

It is easy to check that $\text{Re}(X, \pi)$ is a skew relation algebra. Moreover, it satisfies $1^{\cup\cup} = 1$

For example, we can take $X = \mathbb{Z}$ and $\pi(n) = n + 1$, or $X = \mathbb{Z}_k$ and $\pi(n) = n +_k 1$.

Relation algebras with an invertible constant

Given a relation algebra $\mathbf{A} = (A, \wedge, \vee, ', \perp, \top, \cdot, 1, \smile)$ and an element $\pi \in A$ that satisfies the identities $\pi\pi\smile = 1 = \pi\smile\pi$ (a bijective element), we define the algebra $\mathbf{A}_\pi = (A, \wedge, \vee, ', \perp, \top, \cdot, 1, \cup, \sqcup)$, where $x^\cup = x\smile\pi$ and $x^\sqcup = \pi x\smile$.

It is easy to see that \mathbf{A}_π is (term equivalent to) a skew relation algebra.

Problems: Is every skew relation algebra of the form \mathbf{A}_π ?

Find all minimal skew RAs (cf. Jónsson [1982], Jipsen & Lukács [1994]).

Is the equational theory of skew relation algebras decidable?

The variety of representable skew relation algebras is generated by the algebras $\text{Re}(X, \pi)$ for any set X and π any permutation on X .

This variety is not finitely axiomatizable.

Conclusion

Much is known about representability of relation algebras

The classical theory is a very natural part of residuated lattices with an involution

Exploiting this connection is useful for the theory of involutive FL-algebras

It also provides a new perspective on relation algebras

There are many open problems that make this a fruitful area for new research

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