

# Right-divisible residuated posets, semilattices and residuated Heyting algebras

Peter Jipsen

School of Computational Sciences and  
Center of Excellence in Computation, Algebra and Topology (CECAT)  
Chapman University, Orange, California

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# Outline

- Quasigroups and groups
- Right-residuated binars
- Right-divisible residuated binars and right generalized hoops
- Generalized hoops and GBL-algebras
- Homomorphic images of finite residuated lattices
- Residuated Heyting algebras

## Groups and quasigroups

A **quasigroup**  $(A, \cdot, \backslash, /)$  is a set with 3 binary operations such that for all  $x, y, z \in A$

$$xy = z \iff x = z/y \iff y = x \backslash z$$

i.e., one can solve **all equations** with no repeated variables

Quasigroups form a **variety** defined by the identities

$$(x/y)y = x = xy/y \quad \text{and} \quad y(y \backslash x) = x = y \backslash yx$$

A quasigroup with **associative**  $\cdot$  is **term-equivalent to a group**:

$$1 = y/y \quad \text{and} \quad x^{-1} = (y/y)/x$$

**Hint:**  $xy/z = x((y/z)z)/z = (x(y/z))z/z = x(y/z)$

hence  $x = xy/y = x(y/y)$  so  $x \backslash x = y/y$

# Residuated binars and semigroups

## Definition

A **residuated binar**  $(A, \leq, \cdot, \backslash, /)$  is a poset  $(A, \leq)$  with 3 binary operations such that for all  $x, y, z \in A$

$$xy \leq z \iff x \leq z/y \iff y \leq x \backslash z$$

i.e., one can solve **simple inequalities**

Residuated binars are defined (relative to posets) by the **inequations**

$$(x/y)y \leq x \leq xy/y \quad \text{and} \quad y(y \backslash x) \leq x \leq y \backslash yx$$

**Farulewski 2005:** The universal theory of residuated binars is **decidable**

## Definition

A **residuated semigroup** is an **associative** residuated binar

If the poset is an **antichain**, then any **residuated semigroup is a group!**  
 $\Rightarrow$  residuated semigroups are **generalizations of groups** (replace  $=$  by  $\leq$ )

# Residuated lattices, $\ell$ -groups and GBL-algebras

## Definition

A residuated lattice  $(A, \wedge, \vee, \cdot, \backslash, /, 1)$  is a residuated  $\ell$ -monoid

i.e., a lattice  $(A, \wedge, \vee)$  and a residuated semigroup with unit

E.g.,  $\ell$ -groups are **residuated lattices**:  $x \backslash y = x^{-1}y$  and  $x/y = xy^{-1}$

All nontrivial  $\ell$ -groups are **infinite**; equational theory is **decidable**

## Definition

A residuated lattice is divisible if  $x \leq y \implies x = y(y \backslash x) = (x/y)y$

Also called a generalized Basic Logic algebra (GBL-algebra)

**Open problem:** Is the equational theory of GBL-algebras **decidable**?

# GBL-algebras

- Are a variety that includes all  $\ell$ -groups, all **Heyting algebras** and all **MV-algebras**
- Have **distributive** lattice reducts (like all the above)  
**[J. - Tsinakis 2002]**
- Factor into a product of an  $\ell$ -group and an **integral GBL-algebra** (IGBL) **[Galatos - Tsinakis 2004]**
- All **finite** GBL-algebras are **commutative** and **integral**  
**[J. - Montagna 2006]**
- All **finite** GBL-algebras are poset products of MV-chains  
**[J. - Montagna 2009]**
- **Open problem:** develop a **structure theory** for IGBL-algebras

## Right-residuated binars

GBL-algebras fairly complicated, so consider **simpler** algebras

### Definition

A **right-residuated binar**  $(A, \leq, \cdot, /)$  is a poset  $(A, \leq)$  with 2 binary operations such that for all  $x, y, z \in A$

$$xy \leq z \iff x \leq z/y$$

Therefore  $\cdot, /$  are **order-preserving in the left argument**:

Let  $x \leq y$ . Then  $yz \leq yz \iff y \leq yz/z \implies x \leq yz/z \iff xz \leq yz$

Similarly  $x/z \leq x/z \iff (x/z)z \leq x \implies (x/z)z \leq y \iff x/z \leq y/z$

It would be nice if  $\leq$  is **definable** from the algebraic operations

## Right-divisible residuated binars

**Lemma:** The following are equivalent in any right-residuated binar.

- (i) For all  $x, y$  ( $x \leq y \iff \exists u(x = uy)$ )
- (ii) For all  $x, y$  ( $x \leq y \iff x = (x/y)y$ ) (i.e. divisibility).
- (iii) The identities  $(y/y)x = x$  and  $(y/x)x = (x/y)y$  hold.

**Proof:** (i) $\Rightarrow$ (ii): Suppose  $x \leq y \iff \exists u(x = uy)$  holds.

Assuming  $x \leq y$  one obtains  $uy = x \leq x$  for some  $u$ , hence  $u \leq x/y$ .

Since  $\cdot$  is order preserving in the left argument, we have  $x = uy \leq (x/y)y$ .

The reverse inequality  $(x/y)y \leq x$  holds in any right-residuated binar, so we conclude that  $x \leq y$  implies  $x = (x/y)y$ .

Conversely, if  $x = (x/y)y$  holds, then  $\exists u(x = uy)$ , whence the first condition implies  $x \leq y$ .



(ii) Divisibility iff (iii)  $(y/y)x = x$  and  $(y/x)x = (x/y)y$

(ii) $\Rightarrow$ (i) since we can take  $u = x/y$ .

(ii) $\Rightarrow$ (iii): Assume that  $x \leq y \iff x = (x/y)y$  for all  $x, y$ .

Since  $x \leq x$ , we get  $x = (x/x)x$ .

We always have  $x \leq xy/y$ , hence  $xy \leq (xy/y)y$  holds.

The reverse inequality is also true in general, so  $xy = (xy/y)y$ .

From the assumption it follows that  $xy \leq y$ .

Therefore we have  $x \leq y/y$  as an identity, hence  $x/x \leq y/y$ .

Interchanging  $x, y$  proves  $x/x = y/y$ .

Multiplying by  $x$  on the right we get  $x = (x/x)x = (y/y)x$ .

(ii) Divisibility iff (iii)  $(y/y)x = x$  and  $(y/x)x = (x/y)y$

To prove  $(y/x)x = (x/y)y$ : Recall  $(x/y)y \leq x$ , and use the assumption with  $x$  replaced by  $(x/y)y$  and  $y$  replaced by  $x$  to get  $(x/y)y = ((x/y)y/x)x$ .

As in the proof of the first identity, we have  $xy \leq y$ .

Dividing and multiplying by  $z$  on both sides gives the identity  $(xy/z)z \leq (y/z)z$ .

Now replace  $x$  by  $x/y$  and  $z$  by  $x$  to see that  $((x/y)y/x)x \leq (y/x)x$ .

It follows that  $(x/y)y \leq (y/x)x$ , and interchanging  $x, y$  proves the identity.

(iii) $\Rightarrow$ (ii): Assume the identities  $(y/y)x = x$  and  $(y/x)x = (x/y)y$  hold.

Want to prove: for all  $x, y$  ( $x \leq y \iff x = (x/y)y$ ) (i.e. divisibility)

(iii)  $(y/y)x = x$  and  $(y/x)x = (x/y)y \Rightarrow$  (ii) *Divisibility*

Let  $x \leq y$ . Then  $(y/y)x \leq y$ , hence  $y/y \leq y/x$ .

Multiply by  $x$  on the right to get  $x = (y/y)x \leq (y/x)x = (x/y)y$ . The reverse inequality follows from right-residuation, whence  $x = (x/y)y$ .

Again, assume the two identities of (iii) holds, and let  $x = (x/y)y$ .

By right-residuation we have  $(y/x)x \leq y$ , so we

deduce  $(x/y)y \leq y$  from the second identity.

Since we started with  $x = (x/y)y$ , we conclude that  $x \leq y$ .

## Right-divisible unital residuated binar

The identities for divisibility are  $(y/y)x = x$  and  $(y/x)x = (x/y)y$

So  $y/y$  is a left unit, and the proof of (ii) $\Rightarrow$ (iii) showed  $x \leq y/y$

Hence  $y/y$  is the top element of the poset, denoted by 1

A **right-divisible unital residuated binar** is a residuated binar  $(A, \leq, \cdot, 1, /)$  such that  $x/x = 1$ ,  $1x = x$  and  $(y/x)x = (x/y)y$  hold

The partial order is definable by  $x \leq y \iff x = (x/y)y$

Note that  $(x/y)y$  is a lower bound for any pair of elements  $x, y$  and we always have  $1 \leq 1/x$ .

**Lemma:** In a right-divisible unital binar the partial order is down-directed and the identity  $1/x = 1$  holds. The order is also definable by  $x \leq y \iff y/x = 1$ .

## Right-divisible unital residuated binars

### Theorem

$(A, \cdot, 1, /)$  is a right-divisible unital residuated binar if and only if it satisfies the (quasi)identities  $x/x = 1$   $1x = x$

$$(y/x)x = (x/y)y$$

$$x/y = 1 \text{ and } y/z = 1 \implies x/z = 1$$

$$z/xy = 1 \iff (z/y)/x = 1$$

Note:  $x \leq y$  if and only if  $y/x = 1$ . This is a partial order:

- **reflexive** by  $x/x = 1$
- **antisymmetric** since if  $x/y = 1$  and  $y/x = 1$  then  $x = 1x = 1y = y$
- **transitive** by implication above

**Open problem:** Can the quasiequations be replaced by **identities**?

**Open problem:** Is the (quasi)equational theory **decidable**?

## The right hoop identity

Adding one more identity produces an interesting sub**variety**

In the arithmetic of real numbers (or in any field) the following equation is fundamental to the **simplification of nested fractions**:

$$\frac{x}{z} = \frac{1}{z} \cdot \frac{x}{y} = \frac{x}{zy}$$

In a right-residuated binar this is the ***right hoop identity***:

$$(x/y)/z = x/zy.$$

## Consequences of the right hoop identity

### Theorem

*In a right divisible unital residuated binar the right hoop identity  $x/yz = (x/z)/y$  implies  $x(yz) = (xy)z$ ,  $x1 = x$  and  $x/1 = x$ .*

### Proof.

$$\begin{aligned}x(yz) &= 1(x(yz)) \quad (\text{left unital}) \\&= [(xy)z/(xy)z](x(yz)) \quad \text{since } 1 = x/x \\&= [((xy)z/z)/xy](x(yz)) \quad (\text{right hoop id.}) \\&= [(((xy)z/z)/y)/x](x(yz)) \quad (\text{right hoop id.}) \\&= [((xy)z/yz)/x](x(yz)) \quad (\text{right hoop id.}) \\&= [(xy)z/x(yz)](x(yz)) \quad (\text{right hoop id.}) \\&= [x(yz)/(xy)z]((xy)z) \quad \text{since } (y/x)x = (x/y)y \\&= \text{reverse steps to get } = (xy)z.\end{aligned}$$

Now  $x \leq 1$  implies  $x = (x/1)1$ , hence

$$x1 = ((x/1)1)1 = (x/1)(11) = (x/1)1 = x.$$

Finally  $x/1 = (x/1)1 = (1/x)x = 1x = x$ . □

## Right generalized hoops

### Definitions

A **right generalized hoop**  $(A, \cdot, 1, /)$  is defined by the identities

$$x/x = 1, \quad 1x = x, \quad (x/y)y = (y/x)x \quad \text{and} \quad x/(yz) = (x/z)/y$$

Define the **term-operation**  $x \wedge y = (x/y)y$  and

a **binary relation**  $\leq$  by  $x \leq y \iff x = x \wedge y$

The next lemma shows that  $\wedge$  is a **semilattice** operation

Hence  $\leq$  is a **partial order** on  $A$

Moreover,  $A$  is **right-residuated** with respect to this order

and the left-unit  $1$  is the **top element**



# Properties of right generalized hoops

## Theorem

Let  $A$  be a right generalized hoop. Then

- (i) the term  $x \wedge y = (x/y)y$  is idempotent, commutative and associative,
- (ii)  $\leq$  is a partial order and  $\wedge$  is a meet operation with respect to  $\leq$ ,
- (iii)  $x \leq y \iff y/x = 1$  for all  $x, y \in A$ ,
- (iv)  $xy \leq z \iff x \leq z/y$  for all  $x, y, z \in A$ , and
- (v)  $x \leq 1$  for all  $x \in A$ , i.e.,  $A$  is integral.

## Proof.

(i) The idempotence follows from the first two identities, and commutativity follows from the third. For associativity we calculate

$$\begin{aligned}(x \wedge y) \wedge z &= (((x/y)y)/z)z = (z/(x/y)y)(x/y)y \\ &= ((z/y)/(x/y))(x/y)y \quad (\text{right hoop id.}) \\ &= ((x/y)/(z/y))(z/y)y \quad \text{by assoc. and right-div} \\ &= (x/(z/y)y)(z/y)y \quad (\text{right hoop id.}) \\ &= x \wedge (z \wedge y) = x \wedge (y \wedge z)\end{aligned}$$



## Proof.

(ii) Reflexivity, antisymmetry and transitivity of  $\leq$  and the observation that  $x \wedge y$  is the greatest lower bound of  $x, y$  follow from (i).

(iii)  $x \leq y$  is equivalent to  $x = (x/y)y$  hence  $y/x = y/((x/y)y) = y/((y/x)x) = (y/x)/(y/x) = 1$ , where the third equality uses the right hoop identity. Conversely, if  $y/x = 1$  then  $(x/y)y = (y/x)x = 1x = x$  and we conclude  $x \leq y$ .

(iv) From  $xy \leq z$  we deduce  $z/xy = 1$  by (iii). Hence  $(x/(z/y))(z/y) = ((z/y)/x)x = (z/xy)x = 1x = x$ , or equivalently  $x \leq z/y$ . Conversely, if  $x \leq z/y$  then  $x = (x/(z/y))(z/y) = (z/xy)x$ , so  $xy = (z/xy)xy = (xy/z)z$  which is equivalent to  $xy \leq z$ .

(v) Since  $xy \leq xy$ , (iv) implies  $x \leq xy/y$ . Multiplying by  $y$  gives  $xy \leq (xy/y)y$ , and the reverse inequality also holds by (iv). Hence  $xy = (xy/y)y$ , or equivalently  $xy \leq y$ . A final application of (iv) produces  $x \leq y/y = 1$ . □

## Right generalized hoops and porrimms

There is a 4-element right generalized hoop s.t.  $\cdot$  is **not order-preserving in the right argument**

$\cdot$	0	$a$	$b$	1	$/$	0	$a$	$b$	1
0	0	0	0	0	0	1	0	0	0
$a$	0	$a$	$b$	$a$	$a$	1	1	0	$a$
$b$	0	$a$	$b$	$b$	$b$	1	0	1	$b$
1	0	$a$	$b$	1	1	1	1	1	1

**Partially ordered right-residuated integral monoids** (or porrimms for short) are right-residuated monoids such that the monoid operation is order-preserving in both arguments

They have been studied by **Blok, Raftery [1997]** and **van Alten [1998]**

Results on **porrimms** do not automatically apply to right generalized hoops

# Generalized hoops

## Definition

A **generalized hoop** is an algebra  $(A, \cdot, 1, \backslash, /)$  such that

- $(A, \cdot, 1, /)$  is a right generalized hoop,  $(A, \cdot, 1, \backslash)$  is a left generalized hoop (defined by the mirror-image identities)
- and both these algebras have the same meet operation, i. e., the identity  $(x/y)y = y(y\backslash x)$  holds

**Generalized hoops** were first studied by **Bosbach [1969]**

The name **hoop** was introduced by **Büchi and Owen [1975]**

Generalized hoops are also called **pseudo hoops**

By the preceding theorem, they are **left- and right-residuated**

**Botur, Dvurečenskij and Kowalski [2012]** prove they are **congruence distributive**

## Multiplication distributes over $\wedge$

In a residuated binar, the residuation property implies that  $\cdot$  distributes over any existing joins in each argument. However, this is not true for meets. The following result was proved by **N. Galatos** for **GBL-algebras** but already holds for **generalized hoops**.

### Theorem

*In any generalized hoop  $(x \wedge y)z = xz \wedge yz$  and  $x(y \wedge z) = xy \wedge xz$ .*

### Proof.

From  $xz \leq xz$  it follows that  $x \leq xz/z$ , hence  $xz \leq (xz/z)z$ . Likewise, from  $xz/z \leq xz/z$  we deduce  $(xz/z)z \leq xz$ , therefore  $xz = (xz/z)z$ . Note that  $(x \wedge y)z \leq xz \wedge yz$  always holds since  $\cdot$  is order-preserving.

Now  $xz \wedge yz = (xz/yz)yz = ((xz/z)/y)yz$  (right hoop id.)

$= (y/((xz)/z))(xz/z)z$  by assoc. and divisibility

$= (y/((xz)/z))xz$  by the derived identity

$\leq (y/x)xz = (y \wedge x)z$  since  $x \leq (xz)/z$ . The second identity is similar.

In the last step we made use of the implication  $x \leq y \Rightarrow z/y \leq z/x$  which holds in all residuated binars. □

## Not true for right generalized hoops

The preceding result requires that  $\cdot$  is **order-preserving in the right argument**

Recall the 4-element right generalized hoop from earlier

$\cdot$		0	$a$	$b$	1			0	$a$	$b$	1
0		0	0	0	0			1	0	0	0
$a$		0	$a$	$b$	$a$			1	1	0	$a$
$b$		0	$a$	$b$	$b$			1	0	1	$b$
1		0	$a$	$b$	1			1	1	1	1

$(a \wedge b)a = (a/b)ba = 0ba = 0$  while  $aa \wedge ba = a \wedge a = (a/a)a = 1a = a$ .

## From generalized hoops to GBL-algebras

Generalized hoops have a **simple** equational definition using only  $\cdot, 1, \backslash, /$

The **negative cone** of any  $\ell$ -group is a generalized hoop

Generalized hoops have a  $\wedge$ -**operation**, but **no join** (in general)

However, every finite generalized hoops is a **reduct of an integral GBL-algebra** since a finite meet semilattice with a top element is a lattice

Moreover, **finite** GBL-algebras are **commutative** [J. - Montagna 2006]

### Definition

A **hoop** is a commutative generalized hoop, i.e.  $xy = yx$ .

Hence **finite** generalized hoops are in fact **hoops**

**Open problem:** Is the equational theory of generalized hoops **decidable**?

## Homomorphic images of residuated lattices

An **efficient** way to construct some **homomorphic images**:

A **congruence relation**  $\theta$  of a residuated lattice is determined by the **congruence class**  $[1]_\theta$

Assume this congruence class has a **smallest element**  $c$

Such an element is a **negative central idempotent**

i.e.,  $c \leq 1$ ,  $cc = c$  and  $cx = xc$  for all  $x$

Let  $I_A = \{c \in A : cc = c \leq 1 \text{ and } (\forall x \in A) cx = xc\}$

$(I_A, \cdot, \vee)$  is **dually isomorphic** to the **congruence lattice** of any **finite**  $A$



Define  $Ac = \{xc : x \in A\}$ , and operations  $u \wedge_c v = (u \wedge v)c$ ,

$$u /_c v = (u / v)c \quad u \setminus_c v = (u \setminus v)c$$

## Theorem

Let  $A$  be a residuated lattice and  $c \in I_A$ . Then  $A_c = (Ac, \wedge_c, \vee, \cdot, c, \setminus_c, /_c)$  is a residuated lattice and the map  $h : A \rightarrow Ac$  given by  $h(x) = xc$  is a surjective homomorphism onto  $A_c$ .

If  $\theta$  is the kernel of  $h$  then  $xc$  is the smallest element of  $[x]_\theta$ .

## Proof.

$Ac$  is closed under the operations:  $xc \vee yc = (x \vee y)c$  and  $(xc)(yc) = xyc$  are both in  $Ac$ , and for  $\wedge_c, /_c, \setminus_c$  this holds by construction.

$h$  is surjective, so check that it is a homomorphism, then the homomorphic image is a residuated lattice (homomorphisms preserve identities)

Distributivity of  $\cdot$  over  $\vee$  shows that  $h$  preserves  $\vee$

Centrality and associativity imply that  $h$  preserves  $\cdot$

$h(x) \wedge_c h(y) = (xc \wedge_c yc) = (xc \wedge yc)c \leq (x \wedge y)c = h(x \wedge y)$  since  $c \leq 1$ ,  
 $(x \wedge y)c \leq xc$  and  $(x \wedge y)c \leq yc$  imply  $(x \wedge y)c \leq (xc \wedge yc)c$  □

## Proof.

From  $(x/y)y \leq x$  we get  $(x/y)yz \leq xz$  and hence  $x/y \leq xz/yz$ .

In particular,  $(x/y)c \leq (xc/yc)c$ , which proves  $h(x/y) \leq h(x)/_c h(y)$ .

For the opposite inequality we have  $(xc/yc)yc \leq xc \leq x$ , hence by centrality and idempotence  $(xc/yc)c \leq (x/y)c$ .

Finally,  $h(1) = 1c = c$ , which is the unit of  $A_c$ . □

The theorem works for arbitrary residuated lattices. However in general it does not construct all homomorphic images, only those where the 1-congruence class of the kernel (and hence every congruence class) has a smallest element.

## Corollary

*Let  $A$  be a finite (or complete) residuated lattice and  $B$  any (complete) homomorphic image of  $A$ . Then  $B$  is isomorphic to  $A_c$  where  $c$  is the smallest negative central idempotent of  $A$  that is mapped by the homomorphism to 1 in  $B$ .*

# Hom. images of GBL-algebras and generalized hoops

## Theorem

Let  $A$  be a GBL-algebra with a central idempotent element  $c \in A$ . Then  $A_c$  is isomorphic to the **principal ideal**  $\downarrow c$ , hence  $\wedge_c = \wedge$  and the map  $h(x) = xc$  does not identify any elements of this ideal.

## Proof.

By **divisibility**, if  $x \leq c$  then  $x = (x/c)c$ , and therefore  $x \in A_c$ . Also,  $h(x) = xc = (x/c)cc = x$ , so  $h \upharpoonright_{\downarrow c}$  is the identity map. □

These results also apply to generalized hoops, and in the finite setting describe all homomorphic images

A version for right-generalized hoops is under investigation

## Residuated structures for other logics

**Residuated lattices** have been studied since 1938 by **Dilworth** and **Ward** as abstractions of **ideal lattices of rings**

**Boolean algebras with (residuated) operators** were studied by **Jónsson** and **Tarski** 1951/2

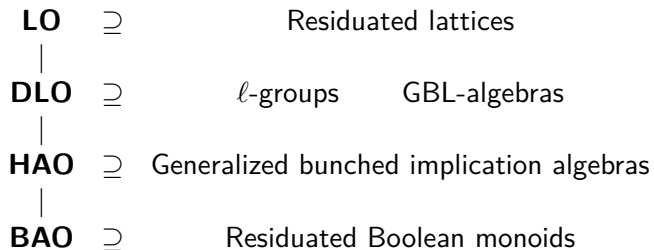
- In logic **implication** plays the role of **residual**
- (generalized) **conjunction** plays the role of **multiplication**

Since the 1980s **residuated lattices** have been viewed as algebraic semantics of **substructural logics**

In computer science **bunched implication logic** was introduced in 1998

with (commutative) **residuated Heyting algebras** as algebraic semantics

# Lattices with operators and some subclasses



## Generalized bunched implication algebras

Recall that a **Heyting algebra** is a residuated lattice with  $\perp$  as bottom element and  $xy = x \wedge y$

In this case we write  $x \rightarrow y$  instead of  $x \backslash y$  ( $= y / x$ )

A **generalized bunched implication algebra** or **GBI-algebra** is an algebra  $(A, \vee, \wedge, \rightarrow, \perp, \cdot, 1, \backslash, /)$  where  $(A, \vee, \wedge, \rightarrow, \perp)$  is a **Heyting algebra**, and  $(A, \vee, \wedge, \cdot, 1, \backslash, /)$  is a **residuated lattice**

### Theorem (Galatos - J. 2016)

*The equational theory of GBI-algebras is **decidable***

*Also true for non-associative GBI-algebras, and for any subvarieties defined by finitely many “simple” identities using only  $\wedge, \vee, \cdot, 1, \top$*

However **divisibility** is **not equivalent** to any such identities

## Residuated Boolean monoids

**BI-algebras** are **commutative** GBI-algebras; also equationally **decidable**

**Applications** in computer science; basis of **separation logic**

If **Heyting algebra** is replaced by **Boolean algebra**, we get **classical GBI-algebras**, also known as **residuated Boolean monoids**.

Theorem (Kurucz et. al. 1995)

*The equational theory of residuated Boolean monoids is **undecidable** (and also for many subvarieties, including classical BI-algebras)*

Moreover, **homomorphic images** of **finite GBI-algebras** can be computed as for residuated lattices

They are **determined** by  $c \in I_A$  with the property that  $\downarrow(\top c) \subseteq Ac$

(since Heyting algebras are GBL-algebras)

Thank you