The categorical equivalence between complete (semi)lattices with operators and contexts with relations

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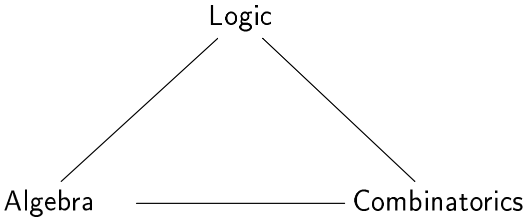
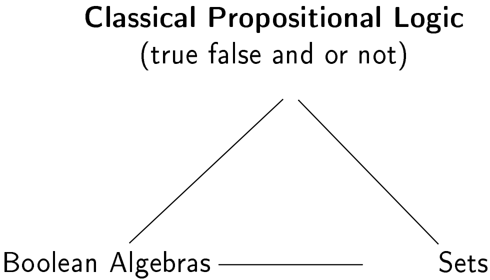
Outline

- Algebraic Logic Overview
- Boolean Algebras with operators $\Longleftrightarrow$ Relational structures
- Equivalence of contexts with complete join (semi)lattices
- The category of Chu-relation morphisms
- Complete lattices with operators $\Longleftrightarrow$ Relational contexts
- Morphisms for relational contexts
- Example: Complete residuated lattices and residuated frames
- Application: Poset products

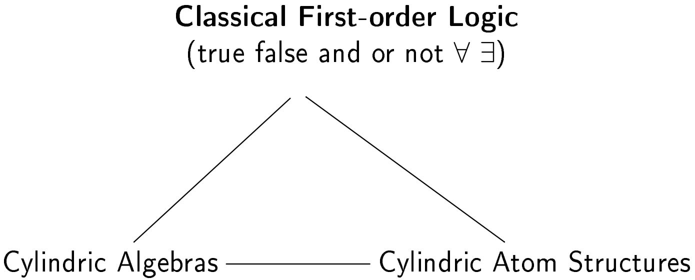
Algebraic Logic



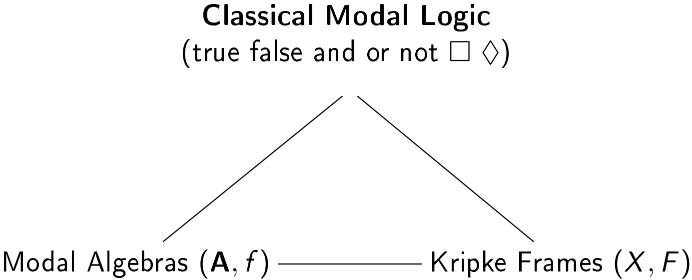
Classical Algebraic Logic



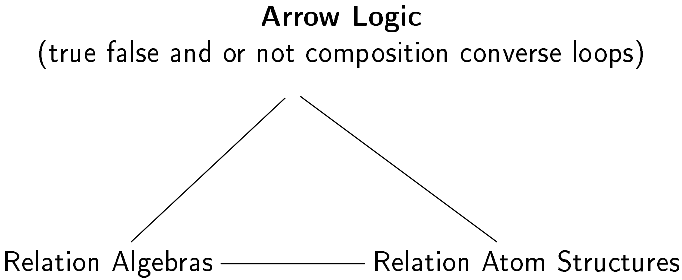
Classical Algebraic Logic



Classical Algebraic Logic



Classical Algebraic Logic



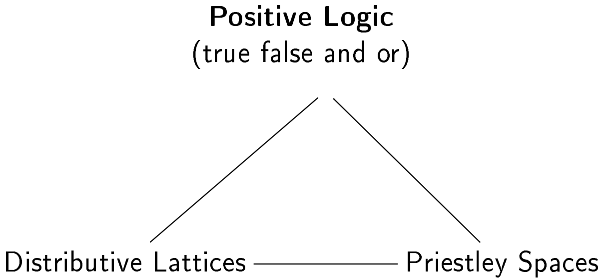
Classical Algebraic Logic

Classical Multi-modal Logic

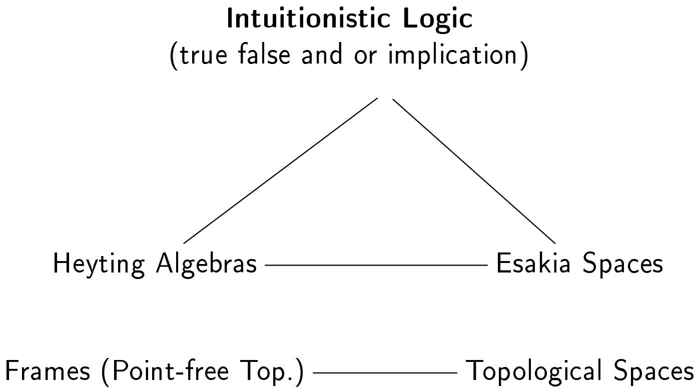
(true false and or not $\Box_i^{(n_i)}$ $\Diamond_i^{(n_i)}$)

BAs with Operators — Relational Structures

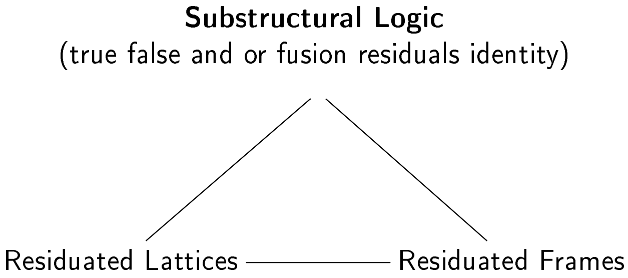
Positive Logic



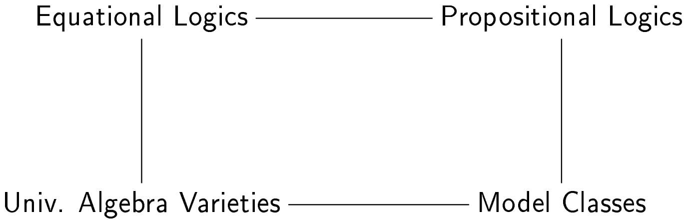
Intuitionistic Logic



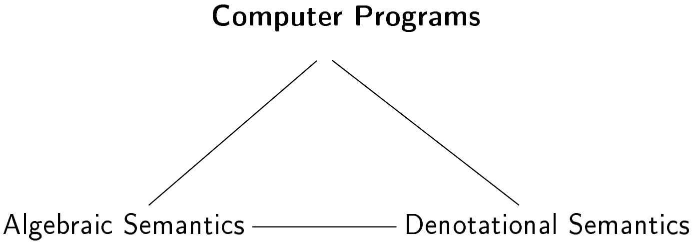
Substructural Logic



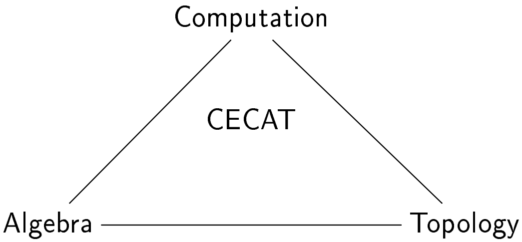
Universal Algebraic Logic



Modern Algebraic Logic



Modern Algebraic Logic



Boolean algebras and Sets

The category **caBA** of complete and atomic Boolean algebras and complete homomorphisms

is dual to the category **Set** of sets and functions:

On objects $B(X) = \mathcal{P}(X)$ and $G(\mathcal{P}(X)) = X$

On morphisms, for $g : X \to Y$ and $h : \mathcal{P}(Y) \to \mathcal{P}(X)$

$B(g)(S) = g^{-1}[S]$ and $G(h)(x)$ is the unique y s. t. $x \in h(\{y\})$

This works because g^{-1} preserves $\bigcap$ and $\bigcup$

Adding complete operators and relations

$\mathbf{caBAO}_\tau$ is the category of caBAs with **completely join-preserving operations** of type $\tau : \mathcal{F} \to \omega$

Each $f \in \mathcal{F}$ has arity $\tau(f)$

The **objects** are $\mathbf{A} = (A, \bigwedge, \bigvee, ', \{f^{\mathbf{A}} : f \in \mathcal{F}\})$

The morphisms are **complete homomorphisms**
i.e., $h : \mathbf{A} \to \mathbf{B}$ preserves all joins, meets, complement, and
$h(f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$

A **central duality of algebraic logic** is that $\mathbf{caBAO}_\tau \equiv^\partial \mathbf{RS}_\tau$

$\mathbf{RS}_\tau$ = category of relational (Kripke) structures
$\mathbf{X} = (X, \{F^{\mathbf{X}} : F \in \mathcal{F}\})$ where $F^{\mathbf{X}} \subseteq X^{\tau(F)+1}$.

But **what** are the morphisms in $\mathbf{RS}_\tau$?

Bounded morphisms

A relation $F^{\mathbf{X}} \subseteq X^{n+1}$ defines an operation $f : \mathcal{P}(X)^n \to \mathcal{P}(X)$ by $f(S_1, \ldots, S_n) = F^{\mathbf{X}}[S_1, \ldots, S_n] = \{z : (x_1, \ldots, x_n, z) \in F^{\mathbf{X}}$ for some $x_i \in S_i\}$

Then f is a $\bigcup$-preserving operation on $B(\mathbf{X})$

From f we can recover $F^{\mathbf{X}}$ by $(x_1, \ldots, x_n, z) \in F^{\mathbf{X}}$ iff $z \in f(\{x_1\}, \ldots, \{x_n\})$

For a function $g : X \to Y$ to be a **morphism** we want $g^{-1}[F^{\mathbf{Y}}[S_1, \ldots, S_n]] = F^{\mathbf{X}}[g^{-1}[S_1], \ldots, g^{-1}[S_n]]$

Since $F[]$ and $g^{-1}[]$ are $\bigcup$-preserving it suffices to check for $S_i = \{y_i\}$

Want $g^{-1}[F^{\mathbf{Y}}[\{y_1\},\ldots,\{y_n\}]] = F^{\mathbf{X}}[g^{-1}[\{y_1\}],\ldots,g^{-1}[\{y_n\}]]$

$x \in g^{-1}[F^{\mathbf{Y}}[\{y_1\},\ldots,\{y_n\}]]$ iff $x \in F^{\mathbf{X}}[g^{-1}[\{y_1\}],\ldots,g^{-1}[\{y_n\}]]$

$g(x) \in F^{\mathbf{Y}}[\{y_1\},\ldots,\{y_n\}]$ iff $\exists x_i \in g^{-1}[\{y_i\}]((x_1,\ldots,x_{n,}x) \in F^{\mathbf{X}})$

$(y_1,\ldots,y_n,g(x)) \in F^{\mathbf{Y}}$ iff $\exists x_i(g(x_i) = y_i$ and $(x_1,\ldots,x_n,x) \in F^{\mathbf{X}})$

g is a **bounded morphism** if it satisfies the above for all $F \in \mathcal{F}$

Back: $(x_1,\ldots,x_{n,}x) \in F^{\mathbf{X}} \Rightarrow (g(x_1),\ldots,g(y_n),g(x)) \in F^{\mathbf{Y}}$ and

Forth: $(y_1,\ldots,y_n,g(x)) \in F^{\mathbf{Y}} \Rightarrow \exists x_i \in X$ such that $g(x_i) = y_i$ and $(x_1,\ldots,x_n,x) \in F^{\mathbf{X}}$

Aim: extend this duality to (complete semi)lattices with operators

Formal Concept Analysis

A ***context*** is a structure $\mathbf{X} = (X_-, X_+, X)$ such that

X_-, X_+ are sets and $X \subseteq X_- \times X_+$.

The ***incidence relation*** X determines two functions
$X^\uparrow : \mathcal{P}(X_-) \to \mathcal{P}(X_+)$ and $X^\downarrow : \mathcal{P}(X_+) \to \mathcal{P}(X_-)$ by

$X^\uparrow A = \{b : \forall a \in A\ aXb\}$ and $X^\downarrow B = \{a : \forall b \in B\ aXb\}$.

Gives a ***Galois connection*** from $\mathcal{P}(X_-)$ to $\mathcal{P}(X_+)$, i.e.,
$A \subseteq X^\downarrow B \iff B \subseteq X^\uparrow A$ for all $A \subseteq X_-$ and $B \subseteq X_+$.

$\mathrm{cl}_-(X) = \{X^\downarrow X^\uparrow A : A \subseteq X_-\}$ and $\mathrm{cl}_+(X) = \{X^\uparrow X^\downarrow B : B \subseteq X_+\}$
are **dually isomorphic complete lattices** with **intersection as meet** and **Galois-closure of union as join**.

Background

Contexts are due to Birkhoff; studied in **Formal Concept Analysis**

Let L be a (bounded) $\bigvee$-semilattice

The **Dedekind-MacNeille context** is $C(L) = (L, L, \leq)$

$\mathrm{cl}_{-}(C(L))$ is the **MacNeille completion** $\bar{L}$ of L

For a finite $\bigvee$-semilattice can take $C(L) = (X_{-}, X_{+}, \leq)$ where X_{-} are the join-irreducibles and X_{+} are the meet-irreducibles

For **complete perfect lattices** results similar to the ones below appear in [Dunn Gehrke Palmigiano 2005] and [Gehrke 2006]

For **complete semilattices** they are due to [Moshier 2011]

For **complete residuated lattices** this is joint work with N. Galatos

Complete lattices with complete homomorphisms form a category

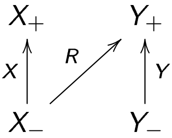
What are the **appropriate** morphisms for contexts?

For a context $\mathbf{X} = (X_-, X_+, X)$ the relation X is an identity morphism that induces the identity map $X^{\downarrow}X^{\uparrow}$ on the closed sets

A **context morphism** $R : \mathbf{X} \to \mathbf{Y} = (Y_-, Y_+, Y)$ is a relation $R \subseteq X_- \times Y_+$ such that $R^{\uparrow}X^{\downarrow}X^{\uparrow} = R^{\uparrow} = Y^{\uparrow}Y^{\downarrow}R^{\uparrow}$ (R is compatible)

Lemma

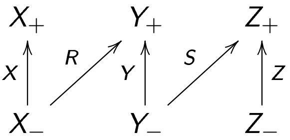
If R is compatible then $Y^{\downarrow}R^{\uparrow} : \mathrm{cl}_-(X) \to \mathrm{cl}_-(Y)$ preserves $\bigvee$



Theorem

(i) The collection **Cxt** *of all contexts with compatible relations as morphisms is a category*

Composition



so $xR;Sy$ *iff* $x \in R^{\downarrow} J^{\uparrow} S^{\downarrow}\{y\}$

(ii) The category **Cxt** *is equivalent to the category* **SUP** *of complete semilattices with completely join-preserving homomorphisms*

The adjoint functors are $\mathrm{cl}_- : \mathbf{Cxt} \to \mathbf{SUP}$ *and* $\mathrm{C} : \mathbf{SUP} \to \mathbf{Cxt}$

On morphisms, $\mathrm{cl}_-(R) = Y^{\downarrow} R^{\uparrow} : \mathrm{cl}_-(X) \to \mathrm{cl}_-(Y)$ *and for a*

SUP *morphism* $h : L \to M$, $C(h) = \{(x, y) \in L \times M : h(x) \leq y\}$

Lattice compatible morphisms

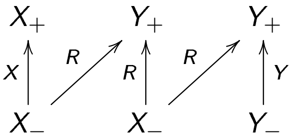
Lemma. *$Y^{\downarrow}R^{\uparrow}$ preserves $\bigwedge$ iff there exists a compatible relation $R_* : Y_- \to X_+$ such that $Y^{\downarrow}R^{\uparrow} = R_*^{\downarrow}X^{\uparrow}$* (call R lattice compatible)

Theorem. *The category* **LCxt** *of all contexts with* ***lattice compatible relations*** *as morphisms is equivalent to the category* **CLat** *of complete lattices with complete lattice morphisms.*

Lemma. *(i) $R : X \to Y$ is a monomorphism in* **Cxt** *iff $R^{\downarrow}R^{\uparrow} = X^{\downarrow}X^{\uparrow}$*

(ii) $R : X \to Y$ is a epimorphism in **Cxt** *iff $R^{\uparrow}R^{\downarrow} = Y^{\uparrow}Y^{\downarrow}$*

Every morphism has **itself** as epi-mono factorization



LCxt is “much larger” than **CLat** since many different contexts represent the same lattice

It is easier to construct examples in **LCxt** than in **CLat**

Contexts can be logarithmic in size compared to their lattices of Galois closed sets

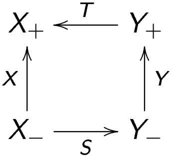
So checking compatibility of R is much more efficient than checking the homomorphism property between large lattices

Now want to extend to complete lattices with operators, relational contexts

Chu-relation morphisms

A *Chu-relation pair* $(S, T) : X \to Y$ is a pair of relations $S \subseteq X_- \times Y_-$ and $T \subseteq Y_+ \times X_+$ such that

$$y \in Y^{\uparrow} S[\{x\}] \iff x \in X^{\downarrow} T[\{y\}]$$



Chu-relation morphisms

Lemma

For a relation $S \subseteq X_- \times Y_-$ *the following are equivalent:*

1. *for all* $A \subseteq X_-$ *we have* $S[X^{\downarrow}X^{\uparrow}A] \subseteq Y^{\downarrow}Y^{\uparrow}S[A]$
2. *for all* $y \in Y^+$, $\{x \in X_- : S[\{x\}] \subseteq Y^{\downarrow}\{y\}\}$ *is in* $cl_-(X)$
3. *there exists a relation* $T \subseteq Y_+ \times X_+$ *such that* (S, T) *are a Chu-relation pair*
4. *the relation* $R = \{(x, y) \in X_- \times Y_+ : y \in Y^{\uparrow}S[\{x\}]\}$ *is a context morphism*

and they imply that the map $Y^{\downarrow}Y^{\uparrow}S[\,] : cl_-(X) \to cl_-(Y)$ *is* $\bigvee$*-preserving.*

1. is the *nuclear condition* from [Galatos-J. *Residuated Frames*]

Contexts with Chu-relation morphisms

Lemma

If $(S, T) : X \to Y$ and $(S', T') : Y \to Z$ are Chu-relation pairs, then $(S; S', T'; T)$ is a Chu-relation pair from X to Z, and (id_{X_-}, id_{X_+}) is a Chu-relation pair from X to X, hence contexts with Chu-relation pairs form a category, called ***RelCxt****.*

This category is not equivalent to **Cxt** since many Chu-relation pairs can correspond to the same context morphism R.

However there are functors F : **Cxt** $\to$ **RelCxt** and G : **RelCxt** $\to$ **Rel** given by $F(R) = (R_0, R_1)$ where $xR_0y \iff y \in Y^{\downarrow}R^{\uparrow}\{x\}$ and $yR_1x \iff x \in X^{\uparrow}R^{\downarrow}\{y\}$

$xG(S, T)y \iff y \in Y^{\uparrow}S[\{x\}]$

$GF(R) = R$ and $FG(S, T)$ is naturally isomorphic to (S, T)

Complete lattices with complete operators

Let $\mathbf{SUP}_\tau$ be the class of $\bigvee$-semilattices with $\bigvee$-preserving operations of type $\tau : \mathcal{F} \to \omega$, where each $f \in \mathcal{F}$ has arity $\tau(f)$

The **objects** are $\mathbf{L} = (L, \bigvee, \{f^{\mathbf{L}} : f \in \mathcal{F}\})$

The morphisms are **complete homomorphisms**
i.e $h : \mathbf{L} \to \mathbf{M}$ preserves all joins and
$h(f^{\mathbf{L}}(a_1, \ldots, a_n)) = f^{\mathbf{M}}(h(a_1), \ldots, h(a_n))$

$\mathbf{Cxt}_\tau$ = category of contexts with relations
$\mathbf{X} = (X_-, X_+, X, \{F^{\mathbf{X}} : F \in \mathcal{F}\})$ where the $F^{\mathbf{X}} \subseteq X_-^{\tau(F)+1}$ satisfy

$F^{\mathbf{X}}[X^{\downarrow}X^{\uparrow}S_1, \ldots, X^{\downarrow}X^{\uparrow}S_n] \subseteq X^{\downarrow}X^{\uparrow}F^{\mathbf{X}}[S_1, \ldots, S_n]$ for all $S_i \subseteq X_-$

Bounded morphisms for contexts

A relation $F^{\mathbf{X}} \subseteq X_{-}^{n+1}$ defines an operation $f : \mathrm{cl}_{-}(X)^n \to \mathrm{cl}_{-}(X)$ by $f(S_1, \ldots, S_n) = X^{\downarrow}X^{\uparrow}F^{\mathbf{X}}[S_1, \ldots, S_n]$

$= X^{\downarrow}X^{\uparrow}\{z : (x_1, \ldots, x_n, z) \in F^{\mathbf{X}}$ for some $x_i \in S_i$, all $i = 1, \ldots, n\}$

Then f is a $\bigvee$-preserving operation on $\mathrm{cl}_{-}(\mathbf{X})$

For a relation $R : X \to Y$ to be a $\mathbf{Cxt}_{\tau}$ **morphism** we want $Y^{\downarrow}R^{\uparrow}f(S_1, \ldots, S_n) = f(Y^{\downarrow}R^{\uparrow}S_1, \ldots, Y^{\downarrow}R^{\uparrow}S_n)$

Since f and $Y^{\downarrow}R^{\uparrow}$ are $\bigvee$-preserving, enough to check for $S_i = \{x_i\}$

$$Y^{\downarrow}R^{\uparrow}X^{\downarrow}X^{\uparrow}F^{\mathbf{X}}[\{x_1\}, \ldots \{x_n\}] = Y^{\downarrow}Y^{\uparrow}F^{\mathbf{Y}}[Y^{\downarrow}R^{\uparrow}\{x_1\}, \ldots Y^{\downarrow}R^{\uparrow}\{x_n\}]$$

$$\implies R^{\uparrow}F^{\mathbf{X}}[\{x_1\}, \ldots, \{x_n\}] = Y^{\uparrow}F^{\mathbf{Y}}[Y^{\downarrow}R^{\uparrow}\{x_1\}, \ldots, Y^{\downarrow}R^{\uparrow}\{x_n\}]$$

This is the **bounded morphism** condition for contexts with relations

Note that the morphism condition is just for *points* of the context

Can give a similar condition for relational context morphisms using Chu-relation pairs

This has *practical* advantages since composition of Chu-relation pairs is **ordinary relation composition** that does not require intermediate closure calculations

Theorem *The category* $\mathbf{Cxt}_\tau$ *of all contexts with* ***bounded compatible relations*** *as morphisms is equivalent to the category* $\mathbf{SUP}_\tau$ *with complete homomorphisms*

This equivalence restricts to the subcategories of $\mathbf{LCxt}_\tau$ *with bounded compatible lattice relations and* $\mathbf{CLat}_\tau$*, i.e., complete lattices with* $\bigvee$*-preserving operators and complete homomorphisms*

Example: Complete residuated lattices

The general theory can also be applied to algebras with quasi-operators, defined in the distributive case by [Gehrke Priestley 2007], and for operations that satisfy Sahlqvist-type equations

$(L, \bigwedge, \bigvee, \cdot, \backslash, /, e)$ is a **complete residuated lattice** if

- $(L, \bigwedge, \bigvee)$ is a complete lattice,
- $(L, \cdot, e)$ is a monoid and
- $x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y$ for all $x, y, z \in L$

It follows that $\cdot$ is completely join-preserving in each argument

cRL is the category of complete residuated lattices with complete homomorphisms

Residuated frames

$(X_-, X_+, X, \circ, \backslash\backslash, //, E)$ is a **unital residuated frame** (or **ru-frame**) if

- (X_-, X_+, X) is a context
- $\circ \subseteq X_-^3, \quad \backslash\backslash \subseteq X_- \times X_+^2, \quad // \subseteq X_+ \times X_- \times X_+$ and $E \subseteq X_-$
- $X^{\uparrow}(x \circ E) = X^{\uparrow}\{x\} = X^{\uparrow}(E \circ x)$
- $x \circ y \subseteq X^{\downarrow}\{z\} \iff y \in X^{\downarrow}(x \backslash\backslash z) \iff x \in X^{\downarrow}(z // y)$

The last two conditions use the notation $x \circ y = \circ[\{x\}, \{y\}]$

The last condition implies $A \cdot B := X^{\downarrow}X^{\uparrow}(A \circ B)$ is a completely join-preserving operation on the lattice of closed sets

An **associative** ru-frame satisfies $X^{\uparrow}((x \circ y) \circ z) = X^{\uparrow}(x \circ (y \circ z))$

Morphisms for ru-frames

Recall a relation $R : X_- \times Y^+$ is a lattice context morphism if it is compatible $R^\uparrow X^\downarrow X^\uparrow = R^\uparrow = Y^\uparrow Y^\downarrow R^\uparrow$ and there exists a compatible relation $R_* : Y_- \times X_+$ such that $Y^\downarrow R^\uparrow = R_*^\downarrow X^\uparrow$

Also want $Y^\downarrow R^\uparrow(A \cdot B) = (Y^\downarrow R^\uparrow A) \cdot (Y^\downarrow R^\uparrow B)$,
$Y^\downarrow R^\uparrow(A \backslash B) = (Y^\downarrow R^\uparrow A) \backslash (Y^\downarrow R^\uparrow B)$ and
$Y^\downarrow R^\uparrow(A/B) = (Y^\downarrow R^\uparrow A)/(Y^\downarrow R^\uparrow B)$ for all $A, B \in \mathrm{cl}_-(\mathbf{X})$

Since $\cdot$ is $\bigvee$-preserving, the first simplifies to
$Y^\downarrow R^\uparrow(\{x_1\} \cdot \{x_2\}) = (Y^\downarrow R^\uparrow \{x_1\}) \cdot (Y^\downarrow R^\uparrow \{x_2\})$,
$R^\uparrow(x_1 \circ x_2) = Y^\uparrow(Y^\downarrow R^\uparrow \{x_1\} \circ Y^\downarrow R^\uparrow \{x_2\})$ and the others reduce to

$Y^\downarrow R^\uparrow(\{x\} \backslash X^\downarrow \{x'\}) = (Y^\downarrow R^\uparrow \{x\}) \backslash R_*^\downarrow \{x'\}$ and
$Y^\downarrow R^\uparrow(X^\downarrow \{x'\}/\{x\}) = R_*^\downarrow \{x'\}/(Y^\downarrow R^\uparrow \{x\})$

Theorem. *With this definition of morphisms, the category of associative unital residuated frames is equivalent to the category of complete residuated lattices.*

In [Galatos, J.] residuated frames are also defined for the case of involutive FL-algebras

Gentzen systems for (involutive) residuated lattices are used to construct (involutive) residuated frames to prove *cut-elimination*, *finite model properties* and *finite embeddability* results for a range of subvarieties

In these applications the contexts are **rarely** separating or reduced, so it is important to have an equivalence with **all** contexts

Extension to all semilattices: A context is ***algebraic*** if $X^{\downarrow}X^{\uparrow}$ preserves all directed unions

From [Hofmann Mislove Stralka 1974] get an equivalence between **all** join-semilattices and algebraic contexts with compatible morphisms s.t. $Y^{\downarrow}R^{\uparrow}$ preserve all directed unions

Can use Hartung's topological contexts to get equivalence for all lattices with operators

Application: Poset products

Products of lattices = disjoint union of contexts with full incidence relation between distinct parts

Ordinal sums = disjoint unions but with "half-full" incidence relation

Poset products of bounded (residuated) lattices are an intermediate concept

Let $(P, \leq)$ be a poset and consider the context $(P, P, \not\leq)$

Given contexts X_p for $p \in P$, let $X_- = \bigcup_P X_{p-}$, $X_+ = \bigcup_P X_{p+}$ and for $x \in X_p$ and $y \in X_q$ define xXy iff $p \not\leq q$

The context X is the *poset sum* of $\{X_p : p \in P\}$, and $\mathrm{cl}_-(X)$ is the *poset product* of the lattices $\mathrm{cl}_-(X_p)$

Poset products of complete residuated lattices $\Longleftrightarrow$ Poset sums of contexts

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Thank You