Categories of algebraic contexts equivalent to idempotent semirings and domain semirings

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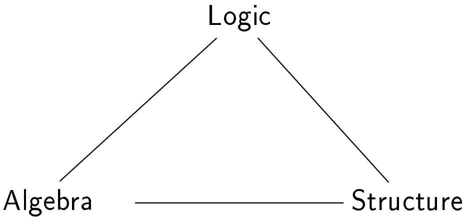
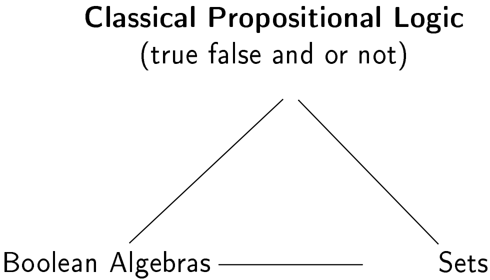
Outline

- Algebraic Logic Overview
- Boolean Algebras with operators $\iff$ Relational structures
- Equivalence of join semilattices with algebraic contexts
- Extend to domain semirings and algebraic contexts with relations
- Morphisms of algebraic contexts with relations
- Example: Describing finite domain semirings
- Application: Constructing contexts of matrix domain semirings

Algebraic Logic



Classical Algebraic Logic



Classical Algebraic Logic

Classical Modal Logic

(true false and or not $\Box$ $\Diamond$)

Modal Algebras $(\mathbf{A}, f)$ —— Kripke Frames (X, F)

Classical Algebraic Logic

Arrow Logic
(true false and or not composition converse loops)

Relation Algebras —— Relation Atom Structures

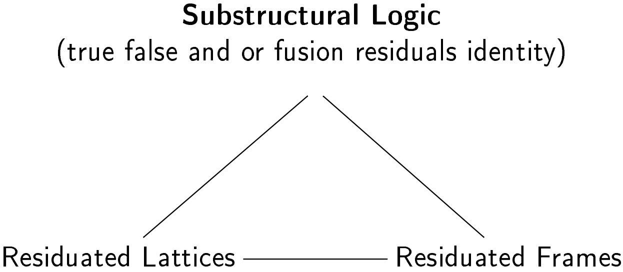
Classical Algebraic Logic

Classical Multi-modal Logic

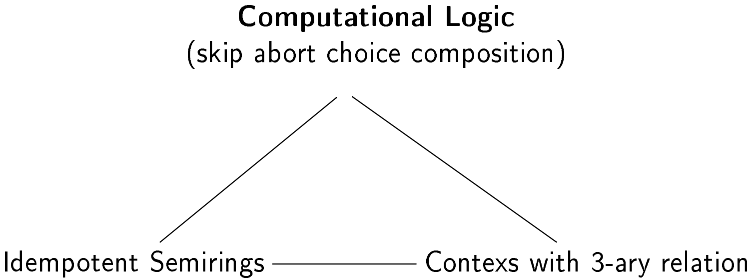
(true false and or not $\Box_i^{(n_i)}$ $\Diamond_i^{(n_i)}$)

BAs with Operators — Relational Structures

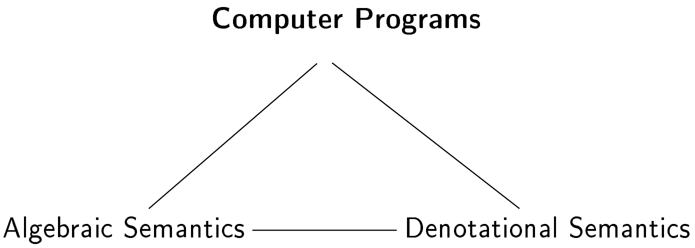
Substructural Logic



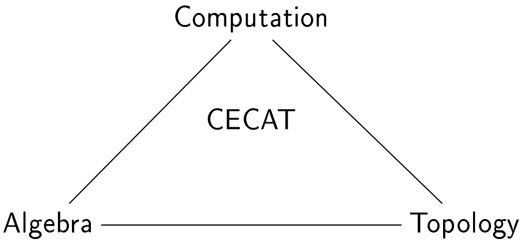
Computational Logic



Modern Algebraic Logic



Modern Algebraic Logic

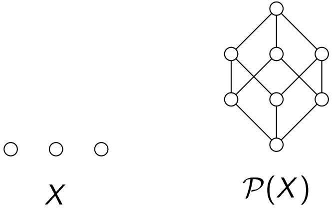


Boolean algebras and Sets

The category **caBA** of complete and atomic Boolean algebras and complete homomorphisms

is dual to the category **Set** of sets and functions:

On objects $B(X) = \mathcal{P}(X)$ and $G(\mathcal{P}(X)) = X$



On morphisms, for $g : X \to Y$ and $h : \mathcal{P}(Y) \to \mathcal{P}(X)$

$B(g)(S) = g^{-1}[S]$ and $G(h)(x)$ is the unique y s. t. $x \in h(\{y\})$

This works because g^{-1} preserves $\bigcap$ and $\bigcup$

Adding complete operators and relations

caBAO$_\tau$ is the category of caBAs with **completely join-preserving operations** of type $\tau : \mathcal{F} \to \omega$

Each $f \in \mathcal{F}$ has arity $\tau(f)$

The **objects** are $\mathbf{A} = (A, \bigwedge, \bigvee, ', \{f^{\mathbf{A}} : f \in \mathcal{F}\})$

The morphisms are **complete homomorphisms**
i.e., $h : \mathbf{A} \to \mathbf{B}$ preserves all joins, meets, complement, and
$h(f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$

A **central duality of algebraic logic** is that **caBAO**$_\tau \equiv^\partial$ **RS**$_\tau$

RS$_\tau$ = category of relational (Kripke) structures
$\mathbb{X} = (X, \{F^{\mathbb{X}} : F \in \mathcal{F}\})$ where $F^{\mathbb{X}} \subseteq X^{\tau(F)+1}$.

But **what** are the morphisms in **RS**$_\tau$?

Bounded morphisms

A relation $F^{\mathbb{X}} \subseteq X^{n+1}$ defines an operation $f : \mathcal{P}(X)^n \to \mathcal{P}(X)$ by $f(S_1, \ldots, S_n) = F^{\mathbb{X}}[S_1, \ldots, S_n] = \{z : (x_1, \ldots, x_n, z) \in F^{\mathbb{X}}$ for some $x_i \in S_i\}$

Then f is a $\bigcup$-preserving operation on $B(\mathbb{X})$

From f we can recover $F^{\mathbb{X}}$ by

$$(x_1, \ldots, x_n, z) \in F^{\mathbb{X}} \iff z \in f(\{x_1\}, \ldots, \{x_n\})$$

For a function $g : X \to Y$ to be a **morphism** we want $g^{-1}[F^{\mathbb{Y}}[S_1, \ldots, S_n]] = F^{\mathbb{X}}[g^{-1}[S_1], \ldots, g^{-1}[S_n]]$

Since $F[]$ and $g^{-1}[]$ are $\bigcup$-preserving it suffices to check for $S_i = \{y_i\}$

Want $g^{-1}[F^{\mathbb{Y}}[\{y_1\},\ldots,\{y_n\}]] = F^{\mathbb{X}}[g^{-1}[\{y_1\}],\ldots,g^{-1}[\{y_n\}]]$

$x \in g^{-1}[F^{\mathbb{Y}}[\{y_1\},\ldots,\{y_n\}]]$ iff $x \in F^{\mathbb{X}}[g^{-1}[\{y_1\}],\ldots,g^{-1}[\{y_n\}]]$

$g(x) \in F^{\mathbb{Y}}[\{y_1\},\ldots,\{y_n\}]$ iff $\exists x_i \in g^{-1}[\{y_i\}]((x_1,\ldots,x_{n,}x) \in F^{\mathbb{X}})$

$(y_1,\ldots,y_n,g(x)) \in F^{\mathbb{Y}}$ iff $\exists x_i(g(x_i) = y_i$ and $(x_1,\ldots,x_n,x) \in F^{\mathbb{X}})$

g is a **bounded morphism** if it satisfies the above for all $F \in \mathcal{F}$

Back: $(x_1,\ldots,x_{n,}x) \in F^{\mathbb{X}} \Rightarrow (g(x_1),\ldots,g(x_n),g(x)) \in F^{\mathbb{Y}}$ and

Forth: $(y_1,\ldots,y_n,g(x)) \in F^{\mathbb{Y}} \Rightarrow \exists x_i \in X$ such that $g(x_i) = y_i$ and $(x_1,\ldots,x_n,x) \in F^{\mathbb{X}}$

Aim: extend this duality to semilattices with operators, including domain semirings

Formal Concept Analysis

A ***context*** is a structure $\mathbb{X} = (X_0, X_1, X)$ such that

X_0, X_1 are sets and $X \subseteq X_0 \times X_1$

The ***incidence relation*** X determines two functions
$X^{\uparrow} : \mathcal{P}(X_0) \to \mathcal{P}(X_1)$ and $X^{\downarrow} : \mathcal{P}(X_1) \to \mathcal{P}(X_0)$ by

$X^{\uparrow}A = \{b : \forall a \in A \; aXb\}$ and $X^{\downarrow}B = \{a : \forall b \in B \; aXb\}$

Gives a ***Galois connection*** from $\mathcal{P}(X_0)$ to $\mathcal{P}(X_1)$, i.e.,
$A \subseteq X^{\downarrow}B \iff B \subseteq X^{\uparrow}A$ for all $A \subseteq X_0$ and $B \subseteq X_1$

$\mathrm{Cl}_0(X) = \{X^{\downarrow}X^{\uparrow}A : A \subseteq X_0\}$ and $\mathrm{Cl}_1(X) = \{X^{\uparrow}X^{\downarrow}B : B \subseteq X_1\}$
are **dually isomorphic complete lattices** with **intersection as meet** and **Galois-closure of union as join**

Background

Contexts are due to Birkhoff; studied in **Formal Concept Analysis**

Let L be a (bounded) $\bigwedge$-semilattice

The **Dedekind-MacNeille context** is $DM(L) = (L, L, \leq)$

$\mathrm{Cl}_0(DM(L))$ is the **MacNeille completion** $\bar{L}$ of L

For a finite $\bigwedge$-semilattice can take $(X_0, X_1, \leq)$ where X_0 are the join-irreducibles and X_1 are the meet-irreducibles

For **complete perfect lattices** a duality with contexts is in [Dunn Gehrke Palmigiano 2005] and [Gehrke 2006]

For **complete semilattices** it is due to [Moshier 2011]

For **complete residuated lattices** this is joint work with N. Galatos

Complete lattices with complete homomorphisms form a category

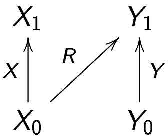
What are the **appropriate** morphisms for contexts?

For a context $\mathbb{X} = (X_0, X_1, X)$ the relation X is an identity morphism that induces the identity map $X^{\downarrow}X^{\uparrow}$ on the closed sets

A **context morphism** $R : \mathbb{X} \to \mathbb{Y} = (Y_0, Y_1, Y)$ is a relation $R \subseteq X_0 \times Y_1$ such that $R^{\uparrow}X^{\downarrow}X^{\uparrow} = R^{\uparrow} = Y^{\uparrow}Y^{\downarrow}R^{\uparrow}$ (R is compatible)

Lemma

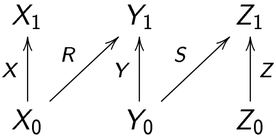
If R is compatible then $R^{\downarrow}Y^{\uparrow} : \mathrm{Cl}_0(Y) \to \mathrm{Cl}_0(X)$ preserves $\bigwedge$



Cxt $\equiv^{\partial}$ Complete meet semilattices

Theorem [Moshier 2011]: (i) The collection **Cxt** of all contexts with compatible relations as morphisms is a category

Composition



so $xR;Sy$ iff $x \in R^{\downarrow}Y^{\uparrow}S^{\downarrow}\{y\}$

(ii) The category **Cxt** is dually equivalent to the category **INF** of complete semilattices with completely meet-preserving homomorphisms

The adjoint functors are $\mathrm{Cl}_0 : \mathbf{Cxt} \to \mathbf{INF}$ and $\mathrm{DM} : \mathbf{INF} \to \mathbf{Cxt}$

On morphisms, $\mathrm{Cl}_0(R) = R^{\downarrow}Y^{\uparrow} : \mathrm{Cl}_0(Y) \to \mathrm{Cl}_0(X)$ and for an

INF morphism $h : L \to M$, $DM(h) = \{(x, y) \in M \times L : x \leq h(y)\}$

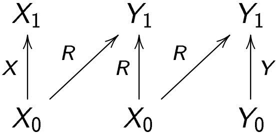
Lattice compatible morphisms

Lemma. *(i)* $R : X \to Y$ *is a monomorphism in* **Cxt** *iff* $R^{\downarrow}R^{\uparrow} = X^{\downarrow}X^{\uparrow}$

(ii) $R : X \to Y$ *is an epimorphism in* **Cxt** *iff* $R^{\uparrow}R^{\downarrow} = Y^{\uparrow}Y^{\downarrow}$

(iii) $R : X \to Y$ *is an isomorphism in* **Cxt** *iff it is both mono and epi iff* $R^{\downarrow}R^{\uparrow}X^{\downarrow} = X^{\downarrow}$ *and* $R^{\uparrow}R^{\downarrow}Y^{\uparrow} = Y^{\uparrow}$

Every morphism has **itself** as epi-mono factorization



Example: Boolean contexts

Let S be any set, and consider the context $\mathbb{S} = (S, S, \neq)$

For any subset A of S we have $\neq^{\uparrow} A = S \setminus A$, so
$\neq^{\downarrow}\neq^{\uparrow} A = S \setminus (S \setminus A) = A$

Hence $\mathsf{Cl}_0(\mathbb{S}) = (\mathcal{P}(S), \bigcap)$

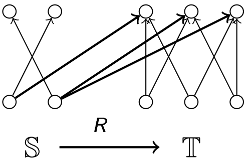
The duality between **INF** and **Cxt** restricts to a duality of complete and atomic Boolean algebras with $\bigwedge$-preserving functions and the category **Rel** of sets and binary relations

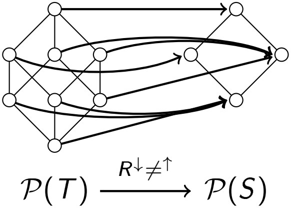
E.g. if $S = \{0, 1\}$ and $T = \{0, 1, 2\}$ then there are $2^{2 \cdot 3} = 64$ binary relations from S to T

Example: Boolean contexts

Therefore there are 64 morphisms from context $\mathbb{S}$ to $\mathbb{T} = (T, T, \neq)$, corresponding to 64 $\bigwedge$-preserving maps from an 8-element Boolean algebra to a 4-element Boolean algebra

Here is one such relation morphism R and its corresponding Boolean homomorphism



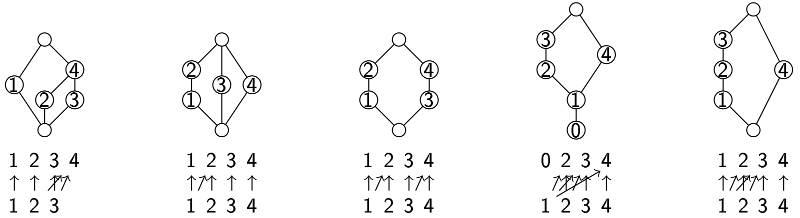


Previous duality can be presented by sets, but for semilattices in general, contexts are required

Note that complete $\bigwedge$-semilattices are complete lattices since the $\bigvee A = \bigwedge\{b : a \leq b \text{ for all } a \in A\}$

If a lattice L is also perfect (e.g. finite), then we can obtain a smallest context by taking $(J_\infty(L), M_\infty(L), \leq)$,

where $J_\infty(L)$ is the set of completely join irreducible elements and $M_\infty(L)$ is the set of completely meet irreducible elements of L



Algebraic contexts

A family $\{A_i : i \in I\}$ of sets is ***directed*** if for all $i, j \in I$ there exists $k \in I$ such that $A_i \cup A_j \subseteq A_k$

For a context X, the closure operator $X^{\downarrow}X^{\uparrow}$ is ***algebraic*** if the closure of any subset is the union of the closures of its finite subsets

Equivalently if for any directed family of sets $\{A_i \subseteq X_0 : i \in I\}$ we have $X^{\downarrow}X^{\uparrow}\bigcup_i A_i = \bigcup_i X^{\downarrow}X^{\uparrow}A_i$

The ***compact*** *sets* of a context X are $K(X) = \{X^{\downarrow}X^{\uparrow}A : A \text{ is a finite subset of } X_0\}$

Algebraic context morphisms

For algebraic contexts X, Y, an ***algebraic*** morphism $R : X \to Y$ is a context morphism such that $R^{\downarrow} Y^{\uparrow}$ preserves directed unions

I.e., for any directed family $\{A_i \subseteq X_0 : i \in I\}$ we have $R^{\downarrow} Y^{\uparrow} \bigcup_i A_i = \bigcup_i R^{\downarrow} Y^{\uparrow} A_i$

This property holds for identity morphisms and is preserved by composition

Hence algebraic contexts form a subcategory of **Cxt** denoted by **ACxt**

All finite contexts are algebraic, and in this case $K(X) = \mathrm{Cl}_0(X)$

Join semilattices with bottom

A ***semilattice*** $(L, +)$ is an algebra where $+$ is associative, commutative and idempotent $(x + x = x)$

$+$ is a join operation if we define $x \leq y$ iff $x + y = y$

0 is a ***bottom element*** if $x + 0 = x$

$\mathbf{JSLat}_0$ is the category of join-semilattices with bottom element 0 and join-preserving homomorphisms that preserve 0, i.e. $h(x + y) = h(x) + h(y)$ and $h(0) = 0$

The context of a join semilattice

An ***ideal*** of a join-semilattice L is a subset D of L such that for all $x, y \in D$, $x + y \in D$ and for all $z \in L$ if $z \leq x$ then $z \in D$

The set of ideals of L is denoted by $I(L)$

Given a join-semilattice L, the ideal context of L is $C(L) = (L, I(L), \in)$

The closure operator $\in^{\downarrow}\in^{\uparrow}$ generates ideals from subsets of L

Since every ideal is the union of its finitely generated subideals, $C(L)$ is an algebraic context.

Theorem. The category $\mathbf{JSLat}_0$ is equivalent to $\mathbf{ACxt}$.

The adjoint functors are
$K : \mathbf{ACxt} \to \mathbf{JSLat}_0, \quad C : \mathbf{JSLat}_0 \to \mathbf{ACxt}$

For a context morphism
$R : X \to Y, \quad K(R) = Y^{\downarrow}R^{\uparrow} : K(X) \to K(Y)$ and

for a $\mathbf{JSLat}_0$ morphism
$h : L \to M, \quad C(h) = \{(a, D) \in L \times I(M) : h(a) \in D\}.$

Proof (outline): Up to isomorphism C and K are inverses on objects of the category:

For L in $\mathbf{JSLat}_0$ and $A \subseteq L$, the closure $\in^{\downarrow}\in^{\uparrow} A = \langle A \rangle$ is the ideal generated by A

For a finite subset A, this ideal is always principal hence $K(C(L))$ is the set of principal ideals of L

$\mathsf{JSLat}_0 \equiv \mathsf{ACxt}$ continued

Any ideal is the union of the principal ideals it contains

Therefore $C(L)$ is algebraic and the map $a \mapsto \downarrow a = \{b \in L : b \leq a\}$ is an isomorphism from L to $K(C(L))$ ordered by inclusion

Now let X be an algebraic context and consider $A, B \in K(X)$

Then $A + B = X^{\downarrow}X^{\uparrow}A_0 + X^{\downarrow}X^{\uparrow}B_0 = X^{\downarrow}X^{\uparrow}(A_0 \cup B_0)$ for some finite $A_0 \subseteq A$ and $B_0 \subseteq B$

Hence $K(X)$ is a semilattice, and the least element is $X^{\downarrow}X^{\uparrow}\emptyset$

We need to prove that $(K(X), I(K(X)), \in)$ is isomorphic to X

$\mathrm{JSLat}_0 \equiv \mathrm{ACxt}$ continued

Define $R : X_0 \to I(K(X))$ by $\qquad xRD \iff X^{\downarrow}X^{\uparrow}\{x\} \in D$

R is an isomorphism since R is compatible, $R^{\downarrow}R^{\uparrow}X^{\downarrow} = X^{\downarrow}$ and $R^{\uparrow}R^{\downarrow}\in^{\uparrow} = \in^{\uparrow}$

Note also that each of these equations holds for all subsets if it is valid for singleton subsets

E.g. $X^{\downarrow}X^{\uparrow}R^{\downarrow} = R^{\downarrow}$: Let D be an ideal of $K(X)$

Then $R^{\downarrow}\{D\} = \{x \in X_0 : X^{\downarrow}X^{\uparrow}\{x\} \in D\} = \{x \in X_0 : x \in A$ for some $A \in D\} = \bigcup D$

Since D is a directed set and X is an algebraic context, $X^{\downarrow}X^{\uparrow}\bigcup D = \bigcup_{A \in D} X^{\downarrow}X^{\uparrow}A = \bigcup D \qquad \square$

In 1974 Hofmann-Mislove-Stralka proved a duality between join-semilattices and algebraic ***lattices*** with maps that preserve all meets and directed joins

However, the category of algebraic contexts is much “bigger” than the category of algebraic lattices

There are many contexts of different sizes that correspond to the same algebraic lattice, e.g., $(B, B, \leq)$ and $(At(B), At(B), \neq)$

So there is more freedom constructing contexts

For many semilattices one can obtain contexts that are logarithmically smaller

Domain semirings

How are join-preserving operations on a semilattice represented on the context side?

We use the example of domain semirings, but it will be clear that the framework can handle semilattices with join-preserving operations of any arity

The equivalence with algebraic contexts is extended to a proper generalization of the duality for $\mathbf{caBA}_\tau$ and $\mathbf{RS}_\tau$

Recall that an ***idempotent semiring*** is an algebra $(L, +, 0, \cdot, 1)$ such that $(L, +, 0)$ is in $\mathbf{JSLat}_0$, $(L, \cdot, 1)$ is a monoid, $\cdot$ is join-preserving in both arguments, and $0x = 0 = x0$

A ***domain semiring*** is of the form $\mathbf{L} = (L, +, 0, \cdot, 1, d)$ such that $(L, +, 0, \cdot, 1)$ is an idempotent semiring, d is join-preserving, $d(0) = 0, \quad d(x) + 1 = 1, \quad d(x)x = x \;$ and $\; d(xd(y)) = d(xy)$

When confusion is unlikely, we usually refer to a domain semiring $\mathbf{L}$ simply by the name of its underlying set L

Contexts with relations

Let X be an algebraic context

To capture the operations of the domain semiring on the semilattice $K(X)$, we need

a ternary relation $\circ \subseteq X_-^3$, a unary relation $E \subseteq X_-$ and a binary relation $D \subseteq X_-^2$

For $A, B \subseteq X_-$, we define

$$A \circ B = \{c \in X_- : (a, b, c) \in \circ \text{ for some } a \in A, b \in B\} \text{ and}$$

$$D[A] = \{b \in X_- : aDb \text{ for some } a \in A\}$$

For $x, y \in X_-$ we further abbreviate $x \circ y = \{x\} \circ \{y\}$ and $D(x) = D[\{x\}]$

Nuclear closure operators

The closure operation $X^{\downarrow}X^{\uparrow}$ is called a *nucleus with respect to* $\circ$ if for all $A, B \subseteq X_{-}$ we have

$$(X^{\downarrow}X^{\uparrow}A) \circ (X^{\downarrow}X^{\uparrow}B) \subseteq X^{\downarrow}X^{\uparrow}(A \circ B)$$

and a *nucleus with respect to* D if for all $A \subseteq X_{-}$ we have

$$D[X^{\downarrow}X^{\uparrow}A] \subseteq X^{\downarrow}X^{\uparrow}D[A]$$

The nucleus property ensures that $X^{\downarrow}X^{\uparrow}(A \circ B)$ and $X^{\downarrow}X^{\uparrow}D[A]$ are join-preserving in each argument

For example:

$$X^{\downarrow}X^{\uparrow}(A \circ \sum_i B_i) = X^{\downarrow}X^{\uparrow}(A \circ X^{\downarrow}X^{\uparrow}\bigcup_i B_i) \subseteq X^{\downarrow}X^{\uparrow}(A \circ \bigcup_i B_i)$$

$$= X^{\downarrow}X^{\uparrow}\bigcup_i (A \circ B_i) = \sum_i (A \circ B_i) \subseteq \sum_i X^{\downarrow}X^{\uparrow}(A \circ B_i)$$

where the first $\subseteq$ follows from the nucleus property, and the reverse inclusion always holds

Idempotent Semiring Contexts

The relations $\circ$ and D are called *algebraic* if for all $A, B \in K(X)$ the operations $X^{\downarrow}X^{\uparrow}(A \circ B)$ and $X^{\downarrow}X^{\uparrow}D[A]$ are also in $K(X)$

An *idempotent semiring context* is of the form $(X_0, X_1, X, \circ, E)$ such that

(X_0, X_1, X) is an algebraic context,

$\circ, E$ are an algebraic ternary and unary relation on X_0,

the closure operator is a nucleus with respect to $\circ$, and for all $x, y, z \in X_0$ we have

$$X^{\uparrow}((x \circ y) \circ z) = X^{\uparrow}(x \circ (y \circ z)) \text{ and}$$

$$X^{\uparrow}(x \circ E) = X^{\uparrow}\{x\} = X^{\uparrow}(E \circ x).$$

Domain Contexts

A *domain context* is a structure $\mathbb{X} = (X_0, X_1, X, \circ, E, D)$ such that

$(X_0, X_1, X, \circ, E)$ is an idempotent semiring context,

the closure operator is also a nucleus with respect to D, and for all $x, y \in X_0$ we have

$$D(x) \subseteq X^{\downarrow} X^{\uparrow} E,$$

$$X^{\uparrow}(D(x) \circ x) = X^{\uparrow}\{x\} \text{ and}$$

$$X^{\uparrow} D[x \circ D(y)] = X^{\uparrow} D[x \circ y]$$

Note that the last 3 conditions need only hold for all elements of X_0, whereas the domain axioms would have to be checked for all elements of the potentially much bigger semilattice of compact sets

Morphisms for domain contexts

Let $\mathbb{X}, \mathbb{Y}$ be two domain contexts

A relation $R \subseteq X_0 \times Y_1$ is a ***domain context morphism*** if it is compatible, algebraic, $R^{\uparrow}(E^{\mathbb{X}}) = Y^{\uparrow}(E^{\mathbb{Y}})$, and for all $A, B \in \mathsf{Cl}_0(\mathbb{X})$ we have

$$R^{\uparrow}(x \circ y) = Y^{\uparrow}(Y^{\downarrow}R^{\uparrow}\{x\} \circ Y^{\downarrow}R^{\uparrow}\{y\}) \text{ and}$$

$$R^{\uparrow}D(x) = Y^{\uparrow}D[Y^{\downarrow}R^{\uparrow}\{x\}]$$

An ***idempotent semiring context morphism*** is defined likewise, but without the last equation

As with *bounded morphisms* (also called *p-morphisms*) in modal logic the notion of domain context morphism can be written as a first-order formula with variables ranging only over elements of the context

The functor K extends to domain contexts by
$K(\mathbb{X}) = (K(X), +, 0, \cdot, 1, d)$ where

$A + B = X^{\downarrow}X^{\uparrow}(A \cup B)$, $\quad 0 = X^{\downarrow}X^{\uparrow}\emptyset$, $\quad A \cdot B = X^{\downarrow}X^{\uparrow}(A \circ B)$,
$1 = X^{\downarrow}X^{\uparrow}E$ and $\quad d(A) = X^{\downarrow}X^{\uparrow}D[A]$

The functor C extends to domain semirings by
$C(\mathbf{L}) = (L, I(L), \in, \circ, \{1\}, D)$ where

$\circ = \{(x, y, z) \in L^3 : x{\cdot}y = z\}$ and $D = \{(x, y) \in L^2 : d(x) = y\}$

Then $K(\mathbb{X})$ is a domain semiring and $C(\mathbf{L})$ is a domain context

E.g. to check that $X^{\downarrow}X^{\uparrow}$ is a nucleus with respect to D, recall that the closure operator generates an ideal from a subset of L

So $D[X^{\downarrow}X^{\uparrow}A] = D[\langle A\rangle] = \{y : d(x) = y \text{ for some } x \in \langle A\rangle\}$

$= \{d(a_1 + \cdots + a_n) : a_i \in A, n \in \mathbb{N}\} = \{d(a_1) + \cdots + d(a_n) : a_i \in A, n \in \mathbb{N}\} \subseteq \langle D[A]\rangle = X^{\downarrow}X^{\uparrow}D[A]$

Categorical equivalences

Theorem: The category **IS** of idempotent semirings is equivalent to the category **ISCxt** of idempotent semiring contexts

The adjoint functors are $K : \mathbf{ISCxt} \to \mathbf{IS}$ and $C : \mathbf{IS} \to \mathbf{ISCxt}$

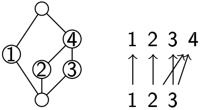
On morphisms, $K(R) = Y^{\downarrow}R^{\uparrow} : K(X) \to K(Y)$ and
$C(h) = \{(a, D) \in L \times I(M) : h(a) \in D\}$

Theorem: Similarly the category **DS** of domain semirings is equivalent to the category **DSCxt** of domain semiring contexts

The adjoint functors are $K : \mathbf{DSCxt} \to \mathbf{DS}$ and $C : \mathbf{DS} \to \mathbf{DSCxt}$ with the operation on morphisms as for idempotent semirings

Applications

Defining a domain semiring by a context, a subset, a binary relation and a ternary relation on the first component of the context

E.g. the (semi)lattice 

can be expanded into 5 nonisomorphic domain semirings where 1 is the identity element

In each case $E = \{1\}$, the binary relation $D = \{(1,1),(2,1),(3,1)\}$, and the 5 ternary relations are

$\circ_1$	2	3
2	$\{2\}$	$\{2\}$
3	$\{2\}$	$\{2\}$

$\circ_2$	2	3
2	$\{2\}$	$\{2\}$
3	$\{2\}$	$\{2,3\}$

$\circ_3$	2	3
2	$\{2,3\}$	$\{2,3\}$
3	$\{2,3\}$	$\{2\}$

$\circ_4$	2	3
2	$\{2,3\}$	$\{2,3\}$
3	$\{2,3\}$	$\{2,3\}$

$\circ_5$	2	3
2	X_0	X_0
3	X_0	X_0

where $1 \circ x = x = x \circ 1$ for all $x \in X_0$

Clearly this is more economical than giving the multiplication tables for five 6-element monoids

Idempotent Matrix Semirings

Given a semiring L, let $M_n(L)$ be the semiring of all $n \times n$ matrices with entries from L

This object has $|L|^{n^2}$ many elements, but for idempotent semirings the context $\mathbb{Y}$ of $M_n(L)$ is much smaller since it can be constructed from n^2 disjoint copies of the idempotent semiring context $\mathbb{X} = C(L)$ as follows:

Let $Y_0 = \{(i, j, a) : a \in X_0, i, j = 1, \ldots, n\}$ and $Y_1 = \{(i, j, a) : a \in X_1, i, j = 1, \ldots, n\}$

Define $(i, j, a)Y(i', j', a') \iff i \neq i'$ or $j \neq j'$ or aXa'

$E = \{(i, i, a) : a \in E, i = 1, \ldots, n\}$, and

$(i, j, a) \circ (k, l, b) = \{(i, l, c) : j = k \text{ and } c \in a \circ b\}$

Then $(Y_0, Y_1, Y, \circ, E)$ is the context of the matrix semiring over L

Conclusion and further research

Join semilattices with bottom are categorically equivalent to algebraic contexts

Kripke frames of idempotent semirings and domain semirings are given by contexts with additional relations

They are similar to atom structures as a tool for constructions and analysis of relation algebras

Can the Kleene-* or $^{\omega}$-operation be represented on the context of an idempotent semiring?

What are the Kripke frames for idempotent semiring with an embedded Boolean test algebra?

Connect the contexts to the coalgebraic view of idempotent semirings

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Thank You