On the structure of balanced residuated partially-ordered monoids

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Residuated partially-ordered monoids (or residuated posets)

A **residuated poset** is a structure $\mathbf{A} = (A, \leqslant, \cdot, 1, /, \backslash)$ such that

- $(A, \leqslant)$ is a poset,
- $(A, \cdot, 1)$ is a monoid,
- $xy \leqslant z \iff x \leqslant z/y \iff y \leqslant x\backslash z$ **(res)**.

$\implies$ $\cdot$ is order-preserving in both arguments

$/, \backslash$ are order-preserving in numerator, order-reversing in denominator

Some examples: $(\mathbb{R}, \leqslant, +, 0, -, -^{\mathrm{op}})$, any ℓ-**group**,

all **groups** (where $\leqslant$ is $=$, $x\backslash y = x^{-1}y$ and $x/y = xy^{-1}$)

all $\wedge, \vee$-free reducts of **residuated lattices**.

The **algebraic semantics of any logic** with a fusion, a truth constant and two implications with a deduction theorem (where $\leqslant$ is $\vdash$)

Balanced residuated posets

A residuated poset is **balanced** if $x/x = x\backslash x$

idempotent if $x^2 = x$ (where $x^2 = x \cdot x$)

integral if $x \leqslant 1$ (i.e. 1 is the top element)

Example: every commutative residuated lattice is balanced,

every Boolean algebra is idempotent and integral

A **residuated lattice** is a residuated poset that is a lattice (has $\vee, \wedge$).

The aim of this talk

Decompose (certain) residuated posets into **simpler components**.

The components are residuated posets with a **unique positive idempotent**.

Reconstruction uses **Płonka sums of metamorphisms**.

Extends structure theory for **even/odd involutive $\mathbf{FL}_e$-chains** [Jenei 2022],

finite commutative idempotent involutive residuated lattices, the components are **Boolean algebras** [Jipsen, Tuyt, Valota 2021],

and a RAMiCS 2022 paper on **locally integral involutive residuated posets**, where the components are **integral** involutive residuated posets [Gil-Férez, Jipsen, Lodhia 2023].

Positive idempotent elements are central

Let $Id^+(\mathbf{A}) = \{p \in A \mid 1 \leqslant p = p^2\}$ = the set of positive idempotent elements of $\mathbf{A}$.

The notation 1_x is an **abbreviation** for x/x.

Lemma

In a residuated poset $\mathbf{A}$ *the following are equivalent:*

- $\mathbf{A}$ *is balanced (i.e.* $x/x = x\backslash x$*),*
- $p \in Id^+(\mathbf{A})$ *implies* p *is central (i.e., for every* x, $px = xp$*),*
- 1_x *is an identity for* x *(i.e.,* $1_x x = x = x 1_x$*).*

Positive idempotents

Lemma

If $\mathbf{A}$ *is a balanced residuated poset then* $(Id^+(\mathbf{A}), \cdot, 1)$ *is a join-semilattice with bottom* 1 *and the order of* $\mathbf{A}$ *agrees with the join-semilattice order on* $Id^+(\mathbf{A})$*.*

Lemma

In a residuated poset $\mathbf{A}$, $Id^+(\mathbf{A}) = \{x/x \mid x \in A\} = \{x\backslash x \mid x \in A\}$*.*

Corollary

A residuated poset satisfies $\forall x, x/x = 1 \iff \forall x, x\backslash x = 1$*.*

An important equivalence relation

Define $x \equiv y \iff 1_x = 1_y$.

Then $\equiv$ is an equivalence relation.

Let $A_x = \{y \in A \mid 1_x = 1_y\}$ be the equivalence classes.

For a residuated poset $\mathbf{A}$ the equivalence classes A_x are called the **local components** of $\mathbf{A}$.

Decomposable residuated posets

A residuated poset is **semilattice decomposable** if

$$1_x = 1_y \implies 1_{x\backslash y} = 1_x.$$

Lemma

A semilattice decomposable residuated poset is **balanced** *and satisfies* $1_x = 1_y \implies 1_{xy} = 1_x$ *and* $1_x = 1_y \implies 1_{x/y} = 1_x$*.*

Examples: All **commutative idempotent** residuated posets.

A RP is **involutive** if $0/(x\backslash 0) = (0/x)\backslash 0$ for some constant 0.

[Gil-Férez, J., Lodhia] all locally integral **involutive** residuated posets are decomposable.

The decomposition theorem

Theorem

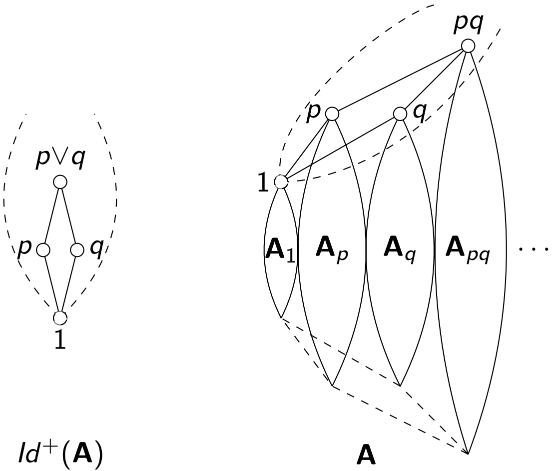
Every decomposable residuated poset $\mathbf{A}$ *is a disjoint union of residuated posets* $\mathbf{A}_p = (A_p, \leqslant_p, \cdot_p, 1_p, \backslash_p, /_p)$ *where*
p *ranges over the join-semilattice* $(Id^+(\mathbf{A}), \leqslant, \cdot)$,
$p = 1_p$ *is the unique positive idempotent of* $\mathbf{A}_p$ *and*
$\leqslant_p, \cdot_p, \backslash_p, /_p$ *are the restrictions of* $\leqslant, \cdot, \backslash, /$ *to* A_p.

Proof.

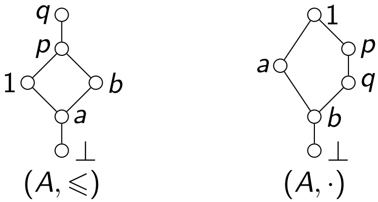
Suppose $\mathbf{A}$ is decomposable and let $p \in Id^+(\mathbf{A})$, so $p = 1_p$.
Then $\mathbf{A}$ is balanced, so p is a left and right identity on A_p.
For $x, y \in A_p$ we have $1_x = p = 1_y$,
so the preceding lemma implies $1_{xy} = 1_x$ and $1_{x/y} = 1_x = p$.
Therefore $xy, x/y \in A_p$, so the $\mathbf{A}_p$ are residuated posets. □

Each component A_p intersects $Id^+(\mathbf{A})$ in a **unique** element $1_x = p$.

The $\mathbf{A}_p$ are **integral** $\iff$ $\mathbf{A}$ is **square-decreasing** $(x \cdot x \leqslant x)$.



A six-element decomposable example



The left poset $(A, \leqslant)$ can be equipped with a commutative idempotent multiplication, i.e., the meet operation of the right poset.

This multiplication preserves all joins of $(A, \leqslant)$, hence is residuated.

This gives a residuated poset **A**, where $Id^+(\mathbf{A}) = \{1, p, q\}$ and $A_1 = \{1, a\}$, $A_p = \{p\}$, and $A_q = \{q, b, \bot\}$.

These sets are closed under residuals, hence **A** is decomposable.

Reconstructing RPs from local components

In many cases the original residuated poset can be reconstructed from the local components and two families of maps:

$\varphi_{pq}, \psi_{pq} : \mathbf{A}_p \to \mathbf{A}_q$ for $p \leqslant q \in Id^+(\mathbf{A})$.

The maps are defined by $\varphi_{pq}(x) = qx$ and $\psi_{pq}(x) = q\backslash x$.

Reconstructing the monoid operation uses a **Płonka sum**.

Reconstructing the order and residuals requires a generalization.

Brief review of Płonka sums

Let $\mathbf{I} = (I, \vee)$ be a join semilattice of **indices**.

A **semilattice directed system** $\Phi = \{\varphi_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j : i \leqslant j \text{ in } \mathbf{I}\}$ is a family of homomorphisms between algebras of the same type if φ_{ii} is the identity on $\mathbf{A}_i$ and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, for all $i \leqslant j \leqslant k$.

If the algebras contain constants, assume $\mathbf{I}$ has a least element $\bot$.

The **Płonka sum** of a semilattice directed system Φ is an algebra $\mathbf{S}$ of the same type defined on the disjoint union of the universes $S = \biguplus_{i \in I} A_i$.

For every n-ary operation symbol σ and $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$,

$$\sigma^{\mathbf{S}}(a_1, \ldots, a_n) = \sigma^{\mathbf{A}_j}(\varphi_{i_1 j}(a_1), \ldots, \varphi_{i_n j}(a_n)),$$

where $j = i_1 \vee \cdots \vee i_n$, and for every constant symbol ω, $\omega^{\mathbf{S}} = \omega^{\mathbf{A}_\bot}$.

Metamorphisms between algebras

Let $\mathbf{A}, \mathbf{B}$ be algebras with operation symbols $\sigma \in \mathcal{O}$.

A **metamorphism** $h : \mathbf{A} \looparrowright \mathbf{B}$ is a sequence of functions $h^\sigma = (h^{\sigma 0}, \dots, h^{\sigma n})$ for each n-ary symbol $\sigma \in \mathcal{O}$ such that

$$h^{\sigma 0}(\sigma^{\mathbf{A}}(a_1, \dots, a_n)) = \sigma^{\mathbf{B}}(h^{\sigma 1}(a_1), \dots, h^{\sigma n}(a_n)).$$

In particular, for every constant ω, $f^\omega = (f^{\omega 0})$ and $f^{\omega 0}(\omega^{\mathbf{A}}) = \omega^{\mathbf{B}}$.

Every homomorphism $g : \mathbf{A} \to \mathbf{B}$ gives rise to a **metamorphism** $h^\sigma = (g, \dots, g)$, and algebras of the same type form a category with componentwise composition of metamorphisms:

If $h \colon \mathbf{A} \looparrowright \mathbf{B}$ and $k \colon \mathbf{B} \looparrowright \mathbf{C}$ then $k \circ h \colon \mathbf{A} \looparrowright \mathbf{C}$ is defined by $(k \circ h)^\sigma = (k^{\sigma 0} \circ h^{\sigma 0}, \dots, k^{\sigma n} \circ h^{\sigma n})$.

The **identity** metamorphism is $id \colon \mathbf{A} \looparrowright \mathbf{A}$, $id^\sigma = (id^A, \dots, id^A)$.

A **semilattice directed system of metamorphisms** is a family $\mathcal{H} = \{h_{pq} \colon \mathbf{A}_p \nrightarrow \mathbf{A}_q : p \leqslant q \text{ in } \mathbf{I}\}$ such that $h_{pp} = id^{\mathbf{A}_p}$ and $h_{qr} \circ h_{pq} = h_{pr}$.

The **Płonka sum of a directed system of metamorphisms** $\mathcal{H}$ is the algebra $\mathbf{A}$ whose universe is $A = \biguplus A_p$, and such that for every n-ary operation σ and all $a_1 \in A_{p_1}, \ldots, a_n \in A_{p_n}$,

$$\sigma^{\mathbf{A}}(a_1, \ldots, a_n) = \sigma^{\mathbf{A}_q}(h^{\sigma 1}_{p_1 q}(a_1), \ldots, h^{\sigma n}_{p_n q}(a_n)),$$

where $q = p_1 \vee \cdots \vee p_n$.

If the type τ contains a constant symbol ω, then we assume that $\mathbf{I}$ has a least element $\perp$ and $\omega^{\mathbf{A}} = \omega^{\mathbf{A}_\perp}$.

Reconstructing Płonka decomposable residuated posets

A residuated poset is **Płonka decomposable** if it satisfies

$$1_{xy} = 1_{x\backslash y} = 1_{x/y} = 1_x \cdot 1_y.$$

For $p \leqslant q \in Id^+(\mathbf{A})$ define $\varphi_{pq}, \psi_{pq} : \mathbf{A}_p \to \mathbf{A}_q$ by

$$\varphi_{pq}(x) = qx \quad \text{and} \quad \psi_{pq}(x) = q\backslash x.$$

Theorem

The algebraic reduct of a Płonka decomposable residuated poset $\mathbf{A}$ *is the Płonka sum of the directed system of metamorphisms* $\mathcal{H} = \{h_{pq} \colon \mathbf{A}_p \nrightarrow \mathbf{A}_q : p \leqslant q \text{ in } \mathbf{I}\}$ *given by*

$$h^1_{pq} = (\varphi_{pq}), \qquad h^{\cdot}_{pq} = (\varphi_{pq}, \varphi_{pq}, \varphi_{pq}),$$

$$h^{\backslash}_{pq} = (\psi_{pq}, \varphi_{pq}, \psi_{pq}), \qquad h^{/}_{pq} = (\psi_{pq}, \psi_{pq}, \varphi_{pq}).$$

Moreover, for all $p, q \in I$, $a \in A_p$, *and* $b \in A_q$,

$$a \leqslant b \quad \iff \quad \varphi_{ps}(a) \leqslant_s \psi_{qs}(b), \quad \text{where } s = pq.$$

Let $\mathbf{I} = (I, \vee)$ be a join-semilattice and (Φ, Ψ) a pair of directed systems of monotone maps $\Phi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant q \text{ in } \mathbf{I}\}$ and $\Psi = \{\psi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant q \text{ in } \mathbf{I}\}$.

Define the relation $\leqslant$ on $A = \biguplus A_p$ as follows: for all $p, q \in I$, $a \in A_p$, and $b \in A_q$,

$$a \leqslant b \quad \iff \quad \varphi_{ps}(a) \leqslant_s \psi_{qs}(b), \quad \text{where } s = p \vee q.$$

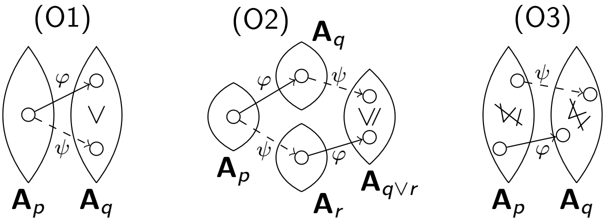
In general, this relation is not a partial order, but it will be if the following three conditions are satisfied. In that case, we call $(A, \leqslant)$ the **sum** of the family of posets $\{\mathbf{A}_p : p \in I\}$ **over** (Φ, Ψ).

(O1) if $p < q$ then $\psi_{pq} <_q \varphi_{pq}$ pointwise,

(O2) if $p \leqslant q, r$ and $t = q \vee r$, then $\varphi_{qt}\psi_{pq} \leqslant_t \psi_{rt}\varphi_{pr}$ pointwise,

(O3) for all $a, b \in A_p$ and $p \leqslant q$, if $\varphi_{pq}(a) \leqslant_q \psi_{pq}(b)$, then $a \leqslant_p b$.

Construction of posets from components



Theorem

Given a pair (Φ, Ψ) of directed systems of monotone maps, the relation $\leqslant$ defined by

$$a \leqslant b \quad \Longleftrightarrow \quad \varphi_{ps}(a) \leqslant_s \psi_{qs}(b), \quad \text{where } s = p \vee q.$$

is a partial order extending the order of each poset **if and only if** (Φ, Ψ) *satisfies* (O1)–(O3).

Construction Theorem

Let $\{\mathbf{A}_p \colon p \in I\}$ be a family of residuated posets indexed on a join semilattice $\mathbf{I} = (I, \vee)$ with least element $\perp$ and Φ, Ψ a pair of semilattice directed systems of monotone maps such that $\mathcal{H} = \{h_{pq} \colon \mathbf{A}_p \looparrowright \mathbf{A}_q : p \leqslant q \text{ in } \mathbf{I}\}$ is a directed system of metamorphisms defined by

$$h^1_{pq} = (\varphi_{pq}), \qquad h^{\cdot}_{pq} = (\varphi_{pq}, \varphi_{pq}, \varphi_{pq}),$$

$$h^{\backslash}_{pq} = (\psi_{pq}, \varphi_{pq}, \psi_{pq}), \qquad h^{/}_{pq} = (\psi_{pq}, \psi_{pq}, \varphi_{pq}).$$

and (Φ, Ψ) satisfies (O1)–(O3). Then the Płonka sum of $\mathcal{H}$ together with the sum of the poset reducts over (Φ, Ψ) is a residuated poset.

Application to involutive residuated posets

An **involutive residuated poset**, or InRP, is a structure of the form $\mathbf{A} = (A, \leqslant, \cdot, 1, \sim, -)$ such that $(A, \leqslant)$ is a poset and $(A, \cdot, 1)$ is a monoid satisfying

$$x \leqslant y \iff x \cdot {\sim} y \leqslant -1 \iff -y \cdot x \leqslant -1. \qquad \text{(ineg)}$$

The monoid operation is residuated with residuals defined by $x \backslash y = {\sim}(-y \cdot x)$ and $x/y = -(y \cdot {\sim} x)$.

An ipo-monoid **A** is **locally integral** if it is balanced, and it satisfies $x \leqslant 1_x$ and $x \backslash 1_x = 1_x$, where $1_x = x/x = -(x \cdot {\sim} x)$.

Płonka sums of InRP

The metamorphisms are determined by the two families of maps

$$\varphi_{pq}(x) = 1_q \cdot x = q \cdot x \text{ and}$$

$$\psi_{pq}(x) = {\sim}(-x \cdot 1_q) = q \backslash x = {\sim}\varphi_{pq}(-x).$$

Hence every locally integral InRP is a Płonka sum, and its order can be recovered by

$$a \leqslant b \quad \Longleftrightarrow \quad \varphi_{ps}(a) \leqslant_s \psi_{qs}(b), \quad \text{where } s = p \vee q.$$

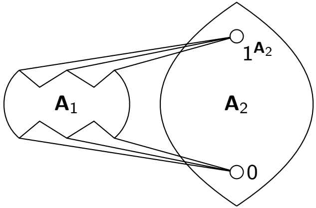
This is the structure theory originally obtained in an ad hoc manner in [Gil-Férez, J., Lodhia 2023].

$$a \leqslant b \quad \Longleftrightarrow \quad \varphi_{ps}(a) \cdot_s \varphi_{ps}({\sim}b) = 0_s, \quad \text{where } s = p \vee q.$$

Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two residuated posets, with two directed systems $\Phi = \{\varphi_{pq} : p \leqslant q\}$ and $\Psi = \{\psi_{pq} : p \leqslant q\}$, indexed over the 2-element chain $1 < 2$, such that the nonidentity maps $\varphi_{12}, \psi_{12} \colon A_1 \to A_2$ are defined by

$$a \mapsto \varphi_{12}(a) = 1^{\mathbf{A}_2} \quad \text{and} \quad a \mapsto \psi_{12}(a) = 0,$$

with 0 a fixed element in A_2 such that $0 < 1^{\mathbf{A}_2}$.



Then (Φ, Ψ) satisfies the conditions of the **Construction Theorem**, hence we obtain a residuated poset $\mathbf{S} = \mathbf{A}_1 \uplus \mathbf{A}_2$.

When will the construction produce a residuated lattice?

In general this is an open problem, but for the two-component Płonka sum it suffices if $\mathbf{A}_1$ is a (chopped) lattice and $\mathbf{A}_2$ is a lattice.

Join and meet are defined as follows:

$$a \vee^{\mathbf{S}} b = \begin{cases} a \vee^{\mathbf{A}} b & \text{if } a, b \in A \text{ have an upper bound in } A \\ 1^{\mathbf{B}} & \text{if } a, b \in A \text{ have no upper bound in } A \\ a \vee^{\mathbf{B}} b & \text{if } a, b \in B \\ a & \text{if } a \in A,\ b \in B \text{ with } b \leqslant 0 \\ b \vee^{\mathbf{B}} 1^{\mathbf{B}} & \text{if } a \in A,\ b \in B \text{ with } b \nleqslant 0, \end{cases}$$

$$a \wedge^{\mathbf{S}} b = \begin{cases} a \wedge^{\mathbf{A}} b & \text{if } a, b \in A \text{ have a lower bound in } A \\ 0 & \text{if } a, b \in A \text{ have no lower bound in } A \\ a \wedge^{\mathbf{B}} b & \text{if } a, b \in B \\ a & \text{if } a \in A,\ b \in B \text{ with } 1^{\mathbf{B}} \leqslant b \\ b \wedge^{\mathbf{B}} 0 & \text{if } a \in A,\ b \in B \text{ with } 1^{\mathbf{B}} \nleqslant b. \end{cases}$$

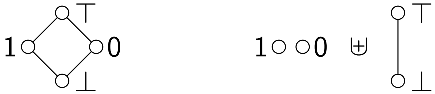
Example of a 4-element relation algebra

For any monoid $\mathbf{M} = (M, \cdot, e)$ the **complex algebra** $\mathcal{P}(\mathbf{M})$ is the residuated lattice $(\mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{e\})$, where for all $X, Y \subseteq M$,

$$X \cdot Y = \{xy \colon x \in X, y \in Y\},$$

$$X\backslash Y = \{z \in M \colon X{\cdot}\{z\} \subseteq Y\} \text{ and } X/Y = \{z \in M \colon \{z\}{\cdot}Y \subseteq X\}.$$

The previous two-component Płonka sum can be used for the 4-element relation algebra $\mathcal{P}(\mathbb{Z}_2)$ where $\mathbb{Z}_2$ is the 2-element group.



$\mathcal{P}(\mathbb{Z}_2)$ $\quad$ $\mathbb{Z}_2$ $\quad$ $\mathbf{2}$

This relation algebra is **not** locally integral, but it is Płonka decomposable, and the Płonka sum decomposition can be applied to all members of the variety of relation algebras generated by $\mathcal{P}(\mathbb{Z}_2)$.

How good is this structure theory?

The table below shows how many Płonka decomposable residuated posets can be built from indecomposable residuated posets (i.e. ones with a unique positive idempotent).

Cardinality $n =$	1	2	3	4	5	6	7	8
Residuated posets (RP)	1	2	5	28	186	1795		
Residuated lattices	1	1	3	20	149	1488	18554	295292
Slat. decomposable RP	1	2	5	24	134	1029		
Płonka decomposable RP	1	2	5	23	121	889		
Unique pos. idem. RP	1	2	4	16	72	516		
Com. idem. Pł. decomp. RP	1	1	2	5	13	36	107	
Idempotent integral RP	1	1	1	2	3	5	8	15

E.g., for commutative idempotent Płonka decomposable residuated posets, **99** seven-element RPs can be constructed from **13** indecomposable components.

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THANKS!