Frames and spaces for distributive quasi relation algebras and involutive FL-algebras

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Relation algebras and atom structures (= Kripke frames)

A **relation algebra A** was defined by Tarski in 1943 as

- a Boolean algebra $(A, \wedge, \vee, \neg, \bot, \top)$,
- a monoid $(A, ;, 1)$ and
- a unary operation $^\smile$ on A such that
- $(a \vee b);c = (a;c) \vee (b;c)$, $(a \vee b)^\smile = a^\smile \vee b^\smile$, $a^{\smile\smile} = a$, $(a;b)^\smile = b^\smile;a^\smile$ and $a^\smile;\neg(a;b) \vee \neg b = \neg b$.

[Maddux 1982] An **atom structure** $(W, R, I, ^\smile)$ is a non-empty set W, $R \subseteq W^3$, $I \subseteq W$, and a map $x \mapsto x^\smile$ on W such that

1. $x = y \iff \exists i \in I\ (Rxiy)$
2. $\exists z\,(Rxyz \text{ and } Rzuv) \iff \exists w\,(Ryuw \text{ and } Rxwv)$
3. $Rxyz \iff Rzy^\smile x$
4. $Rxyz \iff Rx^\smile zy$

Atom structures are **dual to** complete and atomic relation algebras.

Complete and perfect lattices

Recall that a lattice is **complete** if **all** joins and meets exist.

A lattice is **atomic** if every non-bottom element is greater or equal to an atom (i.e., an element that covers $\bot$).

A Boolean algebra is **atomic** $\iff$ every element is a join of atoms.

Tarski duality: a Boolean algebra is complete and atomic $\iff$ it is isomorphic to a powerset Boolean algebra ($\mathbf{caBA} \equiv \mathbf{Set}^{op}$)

A lattice is **perfect** if every element if both the join of completely join-irreducibles and the meet of completely meet-irreducibles.

A distributive lattice is **complete and perfect** $\iff$ it is isomorphic to the **lattice of up-sets of a partial order**.

We now generalize relation algebras to a non-Boolean setting.

Distributive involutive full Lambek algebras

An **involutive residuated lattice** or **InFL-algebra A** is a lattice $(A, \wedge, \vee)$, a monoid $(A, \cdot, 1)$ and two unary operations called **linear negations** $\sim, -$ such that

$$a \cdot b \leq c \iff a \leq -(b \cdot {\sim}c) \iff b \leq {\sim}(-c \cdot a).$$

An InFL-algebra is **cyclic** if ${\sim}a = -a$ and

distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

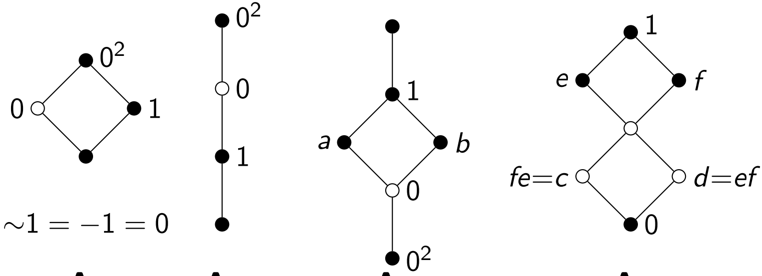
Every **relation algebra is a cyclic distributive InFL-algebra** if we define

$${\sim}a = -a = \neg a^{\smile} \quad \text{and} \quad a\,;b = a \cdot b.$$

Want to define "atom structures" for distributive InFL-algebras.

Examples of DInFL-algebras (and DqRAs)

NB: smallest non-cyclic DInFL-algebra has 7 elements (see $\mathbf{A}_4$ below)



$\mathbf{A}_1$ $\mathbf{A}_2$ $\mathbf{A}_3$ $\mathbf{A}_4$

- Every Sugihara monoid $\mathbf{B} = (B, \wedge, \vee, \cdot, \to, 1, \sim)$ gives rise to a DInFL-algebra $\mathbf{B_D} = (B, \wedge, \vee, \cdot, 1, \sim, -)$ where $- = \sim$.

$\mathbf{A}_3$: $-a = \sim a = b$, $-b = \sim b = a$, $\neg a = a$, $\neg b = b$

$\mathbf{A}_4$: $\sim = -^{-1} : c \to f \to d \to e \to c \quad \neg : c \to e \to c, d \to f \to d$

DInFL-frames

A **DInFL-frame** is a tuple $\mathbb{W} = (W, I, \preceq, R, ^\sim, ^-)$ with

- $W \neq \varnothing$
- I a unary predicate
- $\preceq$ a partial order
- $R \subseteq W^3$
- $^\sim : W \to W$ and $^- : W \to W$
- $I \in \mathrm{Up}(W, \preceq)$.

and:

1. $x \preceq y \iff \exists i \, (i \in I \wedge Rixy)$
2. $x \preceq y \iff \exists i \, (i \in I \wedge Rxiy)$
3. $x \preceq y \wedge Ruvx \implies Ruvy$
4. $\exists z \, (Rxyz \wedge Rzuv) \iff \exists w \, (Ryuw \wedge Rxwv)$
5. $Rxyz^\sim \iff Rzxy^-$
6. $x^{\sim -} \preceq x$ and $x^{- \sim} \preceq x$

Proposition

Let $\mathbb{W} = (W, I, \preceq, R, {}^{\sim}, {}^{-})$ *be a DInFL-frame.*
Let $\mathsf{Up}(W, \preceq)$ *be the set of all upsets of* $(W, \preceq)$.
For all $U, V \in \mathsf{Up}(W, \preceq)$ *define*
$U \circ V = \{w \in W \mid (\exists u \in U)(\exists v \in V)(Ruvw)\}$,
${\sim}U = \{w \in W \mid w^{-} \notin U\}$ *and* $-U = \{w \in W \mid w^{\sim} \notin U\}$.
Then $\mathbb{W}^{+} = (\mathsf{Up}(W, \preceq), \cap, \cup, \circ, I, \sim, -)$ *is a DInFL-algebra.*

Proposition

Let $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -)$ *be a complete perfect DInFL-algebra.*
Let $J^{\infty}(\mathbf{A})$ *be the set of completely join-irreducibles of* $\mathbf{A}$.
Set $I_1 = \{i \in J^{\infty}(\mathbf{A}) \mid i \leqslant 1\}$ *and, for all* $a, b, c \in J^{\infty}(\mathbf{A})$,
define $\preceq = \geqslant$, $R_{\cdot}abc$ *iff* $c \leqslant a \cdot b$, $a^{\sim} = {\sim}\kappa(a)$ *and* $a^{-} = -\kappa(a)$.
Then $\mathbf{A}_{+} = (J^{\infty}(\mathbf{A}), I_1, \preceq, R_{\cdot}, {}^{\sim}, {}^{-})$ *is a DInFL-frame.*

Theorem

If $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -)$ *is a complete perfect DInFL-algebra, then* $\mathbf{A} \cong (\mathbf{A}_+)^+$.

Theorem

If $\mathbb{W} = (W, I, \preceq, R, ^\sim, ^-)$ *is a DInFL- frame, then* $\mathbb{W} \cong (\mathbb{W}^+)_+$.

DInFL-algebras are much more general than relation algebras, and it is not known whether the variety of DInFL-algebras is decidable.

An intermediate variety **DqRA** of **distributive quasi relation algebras** was defined in [Galatos & J 2013].

Distributive quasi relation algebras and DqRA-frames

A **quasi relation algebra (qRA) A** is an InFL-algebra $(A, \wedge, \vee, \cdot, 1, \sim, -)$ with a De Morgan operation $\neg$ such that $\neg\neg x = x$ and

$$\text{(Dm)} \quad \neg(x \wedge y) = \neg x \vee \neg y$$

$$\text{(Dp)} \quad \neg(x \cdot y) = \neg x + \neg y \qquad \text{where } x + y = {\sim}(-y \cdot -x)$$

A **DqRA** is a distributive qRA.

The equational theories of qRA and DqRA are **decidable** .

A **DqRA-frame** is a tuple $\mathbb{W} = (W, I, \preceq, R, {}^{\sim}, {}^{-}, {}^{\neg})$ such that $(W, I, \preceq, R, {}^{\sim}, {}^{-})$ is a DInFL-frame and:

7. $x^{\neg\neg} = x$
8. $x \preceq y \implies y^{\neg} \preceq x^{\neg}$
9. $Rxyz^{-} \iff Ry^{\sim\neg}x^{\sim\neg}z^{\neg}$.

Every RA is a DqRA and every CInFL expands to a qRA

Let $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, ;, 1, ^{\smile})$ be a RA and define $\sim x = \neg x^{\smile}$.

Then $(A, \wedge, \vee, \neg, ;, 1, \sim, \sim)$ is a cyclic DqRA since $\neg\neg x = x$ and (Dm) hold in BA, and (Dp) holds in any RA:

$$\neg x + \neg y = \sim(\sim\neg y; \sim\neg x) = \neg(y^{\smile};x^{\smile})^{\smile} = \neg(x;y).$$

Let $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, \sim)$ be a commutative (i.e., $x \cdot y = y \cdot x$, hence cyclic) InFL-algebra and define $\neg x = \sim x$.

Then $(A, \wedge, \vee, \neg, \cdot, 1, \sim, \sim)$ is a qRA since $\sim\sim x = x$ and (Dm) hold in any cyclic InFL-algebra, and (Dp) holds in any commutative InFL-algebra (by definition of $+$).

Duality for DqRAs and DqRA-frames

Theorem

If $\mathbf{A}$ *is a complete perfect DqRA, then* $\mathbf{A} \cong (\mathbf{A}_+)^+$.

Proof (Outline).

The isomorphism is given by the map $\psi : A \to \mathrm{Up}(J^\infty(\mathbf{A}), \preceq)$ defined by $\psi(a) = \{j \in J^\infty(\mathbf{A}) \mid j \leqslant a\}$. □

Theorem

If $\mathbb{W} = (W, I, \preceq, R, ^\sim, ^-, ^\neg)$ *is a DqRA-frame, then* $\mathbb{W} \cong (\mathbb{W}^+)_+$.

Priestley spaces are duals of bounded distributive lattices

To obtain a duality for incomplete or imperfect distributive lattices, we need to use topology.

A partially ordered topological space $(X, \leqslant, \tau)$ is **totally order-disconnected** if whenever $x \nleqslant y$ there exists a clopen up-set U of X such that $x \in U$ and $y \notin U$.

A **Priestley space** is a compact totally order-disconnected space.

Theorem (Priestley 1970)

- *Every bounded distributive lattice is isomorphic to the set of clopen sets of a Priestley space.*
- *The category of distributive lattices with homomorphisms is dual to the category of Priestley spaces with continuous order-preserving maps.*

Infinite DInFL-algebras/DqRAs may lack bounds. Hence dual spaces must be **doubly-pointed** Priestley spaces.

A **DInFL-space** $(W, I, \preceq, R, ^{\sim}, ^{-}, \bot, \top, \tau)$ is a doubly-pointed DInFL-frame with a compact totally order-disconnected topology τ satisfying:

1. I is clopen.
2. If U and V are clopen proper nonempty up-sets, then $U \circ V$ is clopen.
3. If U is a clopen proper nonempty up-set, then $\sim U$ and $-U$ are clopen.

DqRA-spaces and dualities

A **DqRA-space** $(W, I, \preceq, R, ^{\sim}, ^{-}, ^{\neg}, \bot, \top, \tau)$ is a doubly-pointed DqRA-frame with a compact totally order-disconnected topology τ which satisfies:

1. I is clopen.
2. If U and V are clopen proper nonempty up-sets, then $U \circ V$ is clopen.
3. If U is a clopen proper nonempty up-set, then $\sim U$, $-U$ **and** $\neg U$ **are clopen**.

Here $\neg U = \{w \in W \mid w^{\neg} \notin U\}$.

From algebras to spaces and back

A **generalized prime filter** of a lattice is either a prime filter or the whole lattice or the empty set.

Proposition

For a DInFL-algebra $\mathbf{A}$*, let* $W_{\mathbf{A}}$ *be the set of generalised prime filters of the lattice reduct of* $\mathbf{A}$*.*
For all F, G, H *in* $W_{\mathbf{A}}$*, define* $F \in \mathcal{I}$ *iff* $1 \in F$*,* $F \preceq G$ *iff* $F \subseteq G$*,* $R(F, G, H)$ *iff for all* $a \in F$ *and all* $b \in G$ *we have* $a \cdot b \in H$*,* $F^{\sim} = \{{\sim}a \mid a \notin F\}$ *and* $F^{-} = \{-a \mid a \notin F\}$*. Then the structure* $\mathfrak{F}(\mathbf{A}) = (W_{\mathbf{A}}, \mathcal{I}, \preceq, R, ^{\sim}, ^{-}, \varnothing, A)$ *is a doubly-pointed DInFL-frame.*

Let $\mathfrak{W}(\mathbf{A}) = (\mathfrak{F}(\mathbf{A}), \tau)$ where τ is the topology on the set of generalised prime filters with subbasic open sets of the form $X_a = \{F \in W_{\mathbf{A}} \mid a \in F\}$ and $X_a^c = \{F \in W_{\mathbf{A}} \mid a \notin F\}$.

For a DInFL-space $\mathbb{W}$, we denote by $K_{\mathbb{W}}$ the set of **clopen proper non-empty** upsets of $\mathbb{W}$.

Define $\mathfrak{A}(\mathbb{W})$ to be the algebra $(K_{\mathbb{W}}, \cap, \cup, \circ, I, \sim, -)$.

Dualities

Then $\mathfrak{W}(\mathbf{A})$ is a DInFL-space and $\mathfrak{A}(\mathbb{W})$ is a DInFL-algebra.

Theorem

Let $\mathbf{A}$ be a DInFL-algebra and $\mathbb{W}$ a DInFL-space. Then we have $\mathbf{A} \cong \mathfrak{A}(\mathfrak{W}(\mathbf{A}))$ and $\mathbb{W} \cong \mathfrak{W}(\mathfrak{A}(\mathbb{W}))$.

The same result holds for DqRA and DqRA-spaces if we add $F^{\neg} = \{\neg a \mid a \notin F\}$ to $\mathfrak{W}(\mathbf{A})$ and $\neg U = \{w \in W \mid w^{\neg} \notin U\}$ to $\mathfrak{A}(\mathbb{W})$.

Theorem

- *DInFL-algebras are dual to DInFL-spaces.*
- *DqRAs are dual to DqRA-spaces.*

DqRAs that are subreducts of nonsymmmetric relation algebras

A **proper DqRA** is one that is not a relation algebra (where $x^{\smile} = {\sim}\neg x$).

From a RA we can obtain cyclic DqRAs by letting ${\sim}x = \neg x^{\smile}$, finding $\{\vee, ;, \sim\}$-subreducts, and then checking if $\neg$ can be defined.

Recall that if the subreduct is commutative then $\neg x = {\sim}x$ works.

If the RA is symmetric ($x^{\smile} = x$) then ${\sim}x = \neg x$, so there are no proper subreducts.

Hence we consider only nonsymmetric RAs and check all 16-element RAs for "interesting" subreducts.

In Roger Maddux's book Relation Algebras [2006] there is a list of all nonsymmetric integral relation algebras with 16 elements.

·1	a	r	s
a	1	r	s
r	r	r	⊤
s	s	⊤	s

·2	a	r	s
a	1a	r	s
r	r	r	⊤
s	s	⊤	s

·3	a	r	s
a	1	r	s
r	r	s	1a
s	s	1a	r

·4	a	r	s
a	1a	r	s
r	r	s	1a
s	s	1a	r

·5	a	r	s
a	1	r	s
r	r	rs	⊤
s	s	⊤	rs

·6	a	r	s
a	1a	r	s
r	r	rs	⊤
s	s	⊤	rs

·7	a	r	s
a	1rs	a	a
r	a	r	1rs
s	a	1rs	s

·8	a	r	s
a	⊤	a	a
r	a	r	1rs
s	a	1rs	s

·9	a	r	s
a	1rs	a	a
r	a	s	1
s	a	1	r

·10	a	r	s
a	⊤	a	a
r	a	s	1
s	a	1	r

·11	a	r	s
a	1rs	a	a
r	a	rs	1rs
s	a	1rs	rs

·12	a	r	s
a	⊤	a	a
r	a	rs	1rs
s	a	1rs	rs

·13	a	r	s
a	⊤	ar	a
r	a	r	⊤
s	as	1rs	s

·14	a	r	s
a	1rs	ar	as
r	ar	r	⊤
s	as	⊤	s

·15	a	r	s
a	⊤	ar	as
r	ar	r	⊤
s	as	⊤	s

·16	a	r	s
a	1rs	ar	as
r	ar	rs	⊤
s	as	⊤	rs

·17	a	r	s
a	⊤	ar	as
r	ar	rs	⊤
s	as	⊤	rs

·18	a	r	s
a	1	s	r
r	s	a	1
s	r	1	a

·19	a	r	s
a	1	s	r
r	s	ars	1rs
s	r	1rs	ars

·20	a	r	s
a	1a	rs	rs
r	rs	a	1a
s	rs	1a	a

·21	a	r	s
a	1a	rs	rs
r	rs	ar	⊤
s	rs	⊤	as

·22	a	r	s
a	1a	rs	rs
r	rs	ars	⊤
s	rs	⊤	ars

·23	a	r	s
a	1rs	as	ar
r	as	ar	1rs
s	ar	1rs	as

·24	a	r	s
a	⊤	as	ar
r	as	ar	1rs
s	ar	1rs	as

·25	a	r	s
a	1rs	as	ar
r	as	ars	1rs
s	ar	1rs	ars

·26	a	r	s
a	⊤	as	ar
r	as	ars	1rs
s	ar	1rs	ars

·27	a	r	s
a	1rs	ars	ar
r	as	ar	⊤
s	ars	1rs	as

·28	a	r	s
a	⊤	ars	ar
r	as	ar	⊤
s	ars	1rs	as

·29	a	r	s
a	1rs	ars	ar
r	as	ars	⊤
s	ars	1rs	ars

·30	a	r	s
a	⊤	ars	ar
r	as	ars	⊤
s	ars	1rs	ars

·31	a	r	s
a	⊤	ars	ars
r	ars	a	1a
s	ars	1a	a

·32	a	r	s
a	1rs	ars	ars
r	ars	ar	⊤
s	ars	⊤	as

·33	a	r	s
a	⊤	ars	ars
r	ars	ar	⊤
s	ars	⊤	as

·34	a	r	s
a	1rs	ars	ars
r	ars	as	1a
s	ars	1a	ar

·35	a	r	s
a	⊤	ars	ars
r	ars	as	1a
s	ars	1a	ar

·36	a	r	s
a	1rs	ars	ars
r	ars	ars	⊤
s	ars	⊤	ars

·37	a	r	s
a	⊤	ars	ars
r	ars	ars	⊤
s	ars	⊤	ars

Table: Atom structures (= frames) for the 37 nonsymmetric RAs of cardinality 16. The identity atom 1 is not shown, a string of elements denotes the join of them, and $\sim a = 1rs$, $\sim r = 1ar$, $\sim s = 1as$.

Some background on these 37 nonsymmetric RAs

A relation algebra is **representable** if it is isomorphic to an algebra of binary relation on a set with $\wedge = \cap$, $; = \circ$, $1 = id$, $^{\smile} = {}^{-1}$.

Lyndon [1950] found a nonrepresentable RA, and in [1956] showed that all relation algebras with 8 elements or less are representable.

McKenzie [1966] found the smallest (16-element) nonrepresentable relation algebra (now referred to as 14_{37}).

There are 10 further algebras in the list of 37 that are nonrepresentable: $16_{37}, 21_{37}, 24_{37} - 29_{37}, 32_{37}, 34_{37}$.

The representations of the remaining 26 relation algebras were found by Steven Comer and Roger Maddux.

(Dq)RA-subreducts

We now describe the maximal subreducts of these 37 relation algebras that are proper DqRAs.

When they occur as a subreduct of a representable relation algebra, they are themselves representable (indicated by **bold** names below).

The frames for the first type of subreducts are based on the poset $1{+}1{+}2$, and there are 20 (nonisomorphic) frames of this kind.

The corresponding DqRAs have 12 elements forming the lattice $\mathbf{2}\times\mathbf{2}\times\mathbf{3}$.

$\mathbf{1}_{37}$, $\mathbf{2}_{37}$, $\mathbf{5}_{37}$, $\mathbf{6}_{37}$, $\mathbf{7}_{37}$, $\mathbf{8}_{37}$, $\mathbf{11}_{37}$, $\mathbf{12}_{37}$, 14_{37}, $\mathbf{15}_{37}$, 16_{37}, $\mathbf{17}_{37}$, $\mathbf{20}_{37}$, 21_{37}, $\mathbf{22}_{37}$, $\mathbf{31}_{37}$, 32_{37}, $\mathbf{33}_{37}$, $\mathbf{36}_{36}$, $\mathbf{37}_{37}$.

(Dq)RA-subreducts

In each case the 2-element chain in the frame is given by $s \prec r$ (or isomorphically by $r \prec s$).

To see that this frame corresponds to a subreduct of the listed relation algebras, it suffices to check that $s \leq x \cdot y \implies r \leq x \cdot y$ for all $x, y \in \{a, r, r \vee s\}$.

This formula fails for the other 17 nonsymmetric integral relation algebras with 16 elements (see Table 1).

Hence there are at least 16 representable DqRA with poset 1+1+2 as their frame.

Using the representation game in [J & Šemrl 2023] it has been checked that the DqRA-subreduct of McKenzie's algebra 14_{37} is not representable.

Ten of the remaining 17 relation algebras in the list have a maximal DqRA-subreduct with $1+3$ as poset:

$\mathbf{13}_{37}$, $\mathbf{19}_{37}$, $\mathbf{23}_{37}$, 24_{37}, 25_{37}, 26_{37}, 27_{37}, 28_{37}, 29_{37}, $\mathbf{30}_{37}$.

In this case, the poset of the frame satisfies $s \prec a \prec r$ (or isomorphically $r \prec a \prec s$)

Such a frame corresponds to an 8-element subreduct of a relation algebras if it satisfies $s \leq x \cdot y \implies a \vee r \leq x \cdot y$ and $a \leq x \cdot y \implies r \leq x \cdot y$ for all $x, y \in \{r, a \vee r, a \vee r \vee s\}$.

(Dq)RA-subreducts

The algebras $\mathbf{13}_{37}, 27_{37} - \mathbf{30}_{37}$ are noncommutative.

But the DqRA-subreducts are commutative, hence they can be expanded to DqRAs.

Four of the relation algebras in this list are representable, but the DqRA-subreducts of $\mathbf{19}_{37}$ and $\mathbf{30}_{37}$ are isomorphic, so this gives representations for three 8-element DqRAs.

Other representable 8-element DqRAs can be found as subalgebras of the sixteen representable 12-element DqRAs described above.

7 algebras do not have subreducts that produce proper DqRAs:
$\mathbf{3}_{37}, \mathbf{4}_{37}, \mathbf{9}_{37}, \mathbf{10}_{37}, \mathbf{18}_{37}, 34_{37}, \mathbf{35}_{37}$.

Conclusion

We obtained first-order axiomatizations for DInFL-frames and DqRA-frames that are dual to complete perfect distributive involutive FL-algebras and distributive quasi relation algebras respectively.

Adding doubly-pointed Priestley-space topologies to these frames we obtain dual spaces for these algebras without requiring them to be complete and perfect.

For small nonsymmetric relation algebras, DqRA-frames can be used to provide representations for 16 DqRAs with 12 elements and for 3 DqRAs with 8 elements.

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