Enumerating finite structures

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Outline

- Background
- Semigroups, monoids, semilattices, lattices, semirings,...
- Online mathematical structures database
- Other related packages
- Basic algorithm
- Constraint satisfaction
- Graphs and binary relations; binary decision diagrams
- Relation algebras and ternary relational structures
- Theorem provers

Background

Classical mathematics considers specific structures

e.g. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and functions defined on cartesian products of these.

20th century mathematics emphasized the *axiomatic approach* (originally developed in Greek geometry)

Category theory was developed as a framework for mathematics

Relates well to *object-oriented software development*

Mathematical objects

Sage contains categories and many concrete mathematical objects

Databases and iterators produce streams of objects to test conjectures

Combinatorics mostly studies objects that have finite representations

Enumerative combinatorics aims to count objects of a given size

Classification theorems are used to categorize objects

Structure theorems describe the structure of objects in a given class

First-order structures

Relational structures

Definition: A set with *n*-ary relations defined on it (of fixed arity).

E.g. sets, graphs, digraphs, trees, partially ordered sets, quasiordered sets, 3-hypergraphs, ...

Algebraic structures = universal algebras

Definition: A set with *n*-ary operations defined on it (of fixed arity).

E.g. unary algebras, automata, semigroups, monoids, groups, quasigroups, loops, semilattices, lattices, boolean algebras, semirings, rings, modal algebras, residuated lattices, Kleene algebras...

Relational and algebraic structures

Ordered algebraic structures

E.g. partially ordered semigroups, po-monoids, po-groups, semilattices, lattices, boolean algebras, modal algebras, residuated lattices, ...

Multi-sorted structures

E.g. Vector spaces, modules, G-sets, categories, monads, datatypes, ...

Each of these classes has been studied extensively.

Some online mathematical structures databases

Neil Sloane's *Online Encyclopedia of Integer Sequences*

stores enumerative information about object, but not object themselves

GAP databases of groups (optional Sage package), GUAVA (coding theory), SONATA (nearring library), LOOPS (includes library of loops)

John Halleck's list of modal logic systems (no semantic objects) at

`www.cc.utah.edu/~nahaj/logic/structures/systems`

Mathematical structures list of mostly algebraic classes (few objects) at

`math.chapman.edu/cgi-bin/structures`

Designs at `designtheory.org/database/`

Graph databases and many combinatorial objects are already in Sage

Relational Structure Format

The Sage RelStructure Class

Finite (universal) algebras and relational structures in SAGE are implemented as lists of operation tables and relations.

The base set for a relational structure of size n is 0,1,...,n-1.

Binary operations are *$n \times n$ matrices* with entries from $[0..n-1]$

Binary relations are *dictionaries* with $i : [..., j, ...]$ iff $i \sim j$

Search routines

The file relstruct.py contains code to search for *nonisomorphic* groupoids and binary relations

Searches are done in a simple-minded way by *backtracking*

Search for small algebras

Find all nonisomorphic algebras of a given size

```
find_groupoids(3)                # nonisomorphic binary operatio
find_semigroups(5)               # associative groupoids
find_commutative_semigroups(5)
find_bands(6)                    # idempotent semigroups
find_monoids(6)                  # semigroups with identity
find_commutative_monoids(7)
find_semilattices(7)             # commutative bands
find_lattices(8)                 # semilattices with identity

find_algebras_test(True) # test all find_ methods and compare
                         # lengths of lists with sloane_find
```

Search for small relational structures

Find all nonisomorphic binary relations of a given size.

```
find_binary_relations(5) # nonisomorphic binary relations

find_simple_digraphs(6)  # binary relations without loops

find_simple_graphs(7)    # symmetric relations without loops

find_relations_test(True)# test all find_methods and compare
                         # lengths of lists with sloane_find
```

The search algorithm

A *partial operation table* is a matrix with entries from $[-1, 0, .., n-1]$

-1 denotes an *undefined* entry

```
def complete_operation(m,i,j,associative=False,commutative=Fal
    """
    find next i,j where alg[i][j] = -1 = undefined
    for each val in [0..n-1]
        set alg[i][j] = val
        check_associativity(m)   [if associative=True]
        check_permutations
        if ok, complete_operation(m,i,j+1)
        restore alg
    """
```

The search algorithm

```
def check_associativity(m):
    """
    Return False if the partial operation table  m  violates
        associativity
    Return True otherwise
    """

def check_permutations(m):
    """
    Return False if every completion of some isomorphic
        copy q(m) of m will be lexicographically smaller than
    Return True otherwise
    """
```

To do

Adapt N.I.C.E. code to find *automorphism group* of the partial algebra

Write the search routine as a *Python generator*

Create proper Sage objects from the output

E.g. the monoids should be objects in the *Monoid category*

Discussion

The current approach is simple, lightweight

Reasonably fast for *small* structures

Easily adapted to other classes with more operations or relations

More general approach: *Constraint satisfaction*

More efficient for datastructure for graphs and binary relations: *binary decision diagrams*

Examples needing *ternary* relational structures: relation algebras

On the syntactic side, would like to integrate *logical reasoning* and *theorem provers* in Sage

Residuated lattices

Definition

A *residuated lattice* is a system $(L, \wedge, \vee, \cdot, \backslash, /, e)$ where

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, e)$ is a monoid,
- $\backslash$ and $/$ are binary operations such that the *residuation property* holds:

$$x \cdot y \leq z \quad \text{iff} \quad y \leq x\backslash z \quad \text{iff} \quad x \leq z/y$$

Galatos, J., Kowalski, Ono, (2007) "Residuated Lattices: An algebraic glimpse at substructural logics", Studies in Logic, Elsevier, xxi+509 pp.

The symbol $\cdot$ is often omitted

Properties of residuated lattices

Definition

A residuated lattice is:

- *commutative* if it satisfies $xy = yx$
- *integral* if it satisfies $x \leq e$
- *divisible* if $x \leq y$ implies $x = y(y\backslash x) = (x/y)y$
- *representable* if it is isomorphic to a subdirect product of totally ordered residuated lattices
- *bounded* if it has a mimimum element, and there is an additional constant 0 which denotes this minimum

In a commutative residuated lattice the operations $x\backslash y$ and y/x coincide

Subclasses of residuated lattices

Definition

A *GBL-algebra* is a divisible residuated lattice

A *BL-algebra* is a bounded commutative integral representable GBL-algebra

A *GMV-algebra* is a GBL-algebra satisfying $x \leq y$ implies $y = x/(y\backslash x) = (x/y)\backslash x$

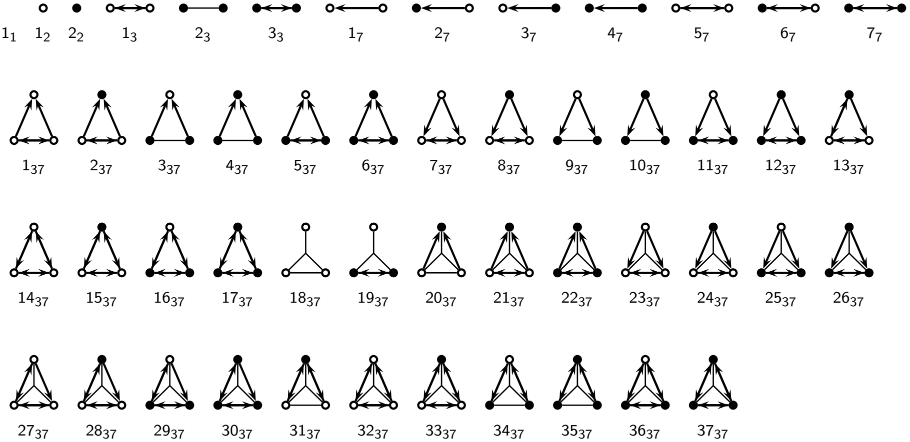
An *MV-algebra* is a bounded commutative GMV-algebra

A *lattice ordered group* or *ℓ-group* is (term-equivalent to) a residuated lattice satisfying $x(x\backslash e) = e$

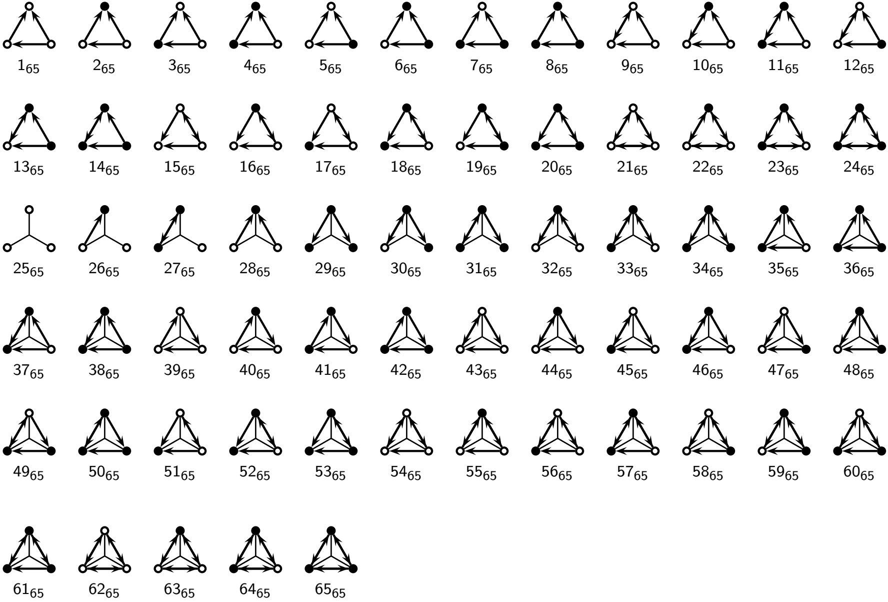
Commutative GMV-algebas (and MV-algebras) are always representable

BL-algebras (Hájek '98) give algebraic semantics of Basic (fuzzy) Logic

The first 50 integral relation algebras

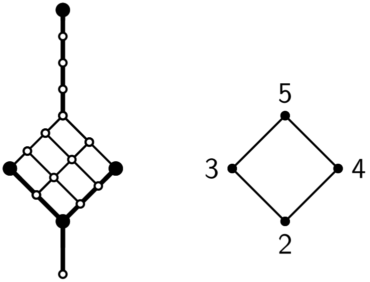


The 65 symmetric integral relation algebras of size 16



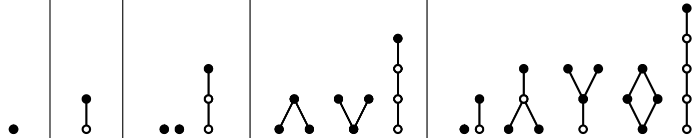
A representation useful for constructing and counting finite GBL-algebras

For example, consider the following lattice structure of a GBL-algebra with 17 elements that is obtained from a poset sum of a 2, 3, 4, and 5-element Wajsberg chain over the poset $\mathbf{2} \times \mathbf{2}$ (the join irreducible idempotents are denoted by black dots)



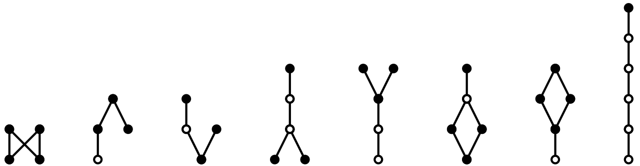
By the previous theorem the same lattice supports $2^6 = 64$ nonisomorphic GBL-algebras since six other join irreducibles could be idempotents

Posets of join-irreducibles of distributive lattices of size n with number of nonisomorphic GBL-algebras below each poset



1	2	3	$n = 4$	$n = 5$	$n = 6$
$\emptyset$ 1	1	2	1 4	1 1 8	2 2 2 1 16

$n = 7$



1 2 2 4 4 2 2 32

A finite GBL-algebra is subdirectly irreducible iff the poset of join-irreducibles has a top element

It is representable (and hence expands to a finite BL-algebra) iff the poset of join-irreducibles is a forest.

Since subdirectly irreducible BL-algebras are chains, it follows that for $n > 1$ there are precisely 2^{n-2} nonisomorphic subdirectly irreducible n-element BL-algebras.

Size $n =$	1	2	3	4	5	6	7
GBL-algebras	1	1	2	5	10	23	49
si GBL-algebras	0	1	2	4	9	19	42
BL-algebras	1	1	2	5	9	20	38
si BL-algebras	0	1	2	4	8	16	32