Classical and Intuitionistic Relation Algebras

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Outline

- Classical relation algebras
- Involutive residuated lattices
- Generalized bunched implication algebras
- Weakening relations and intuitionistic relation algebras
- Representable weakening relation algebras (RwRA)
- Groupoid semantics for RwRA

Classical algebras of binary relations

The **calculus of binary relations** was developed by

A. De Morgan [1864], **C. S. Peirce** [1883], and **E. Schröder** [1895]

At the time it was considered one of the cornerstones of mathematical logic

Alfred Tarski [1941] gave a set of axioms, refined in 1943 to 10 equational axioms, for (abstract) relation algebras

Jónsson-Tarski [1948]: A relation algebra (RA) A is a Boolean algebra with a binary associative operator ; such that: ; has a unit element 1, $x^{\smile\smile} = x$, $(xy)^{\smile} = y^{\smile}x^{\smile}$ and $x^{\smile};\neg(x;y) \leq \neg y$


Independence of Tarski's 10 axioms

(R1) $x \vee y = y \vee x$
(R2) $x \vee (y \vee z) = (x \vee y) \vee z$
(R3) $\neg(\neg x \vee y) \vee \neg(\neg x \vee \neg y) = x$
(R4) $x;(y;z) = (x;y);z$
(R5) $x;1 = x$
(R6) $x^{\smile\smile} = x$
(R7) $(xy)^{\smile} = y^{\smile}x^{\smile}$
(R8) $(x \vee y);z = x;z \vee y;z$
(R9) $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$
(R10) $x^{\smile};\neg(x;y) \vee \neg y = \neg y$

Joint work with **H. Andreka, S. Givant** and **I. Nemeti** [to appear]

For each (Ri) show that (Ri) does not follow from the other identities

McKinsey [early 1940s] showed the independence of (R4)

Need to find an algebra A_i where (Ri) fails and the other identities hold

For example: for (R1) define $A_1 = (\{1, a\}, \vee, \neg, ;, ^{\smile}, 1)$ where 1 is an identity for ; distinct from a, $a;a = 1$, $x^{\smile} = \neg x = x$, and $x \vee y = x$

Check that (R1) fails: $1 \vee a = 1 \neq a = a \vee 1$ and (R2-R10) hold in A_1

Summary of other independence models

$A_2 = \{-1, 0, 1\}$ where $x \vee y = \min(\max(x + y, 1), -1)$ truncated addition

$-$ is subtraction, ; is multiplication, $x^\smile = x$ and $1 = 1$

(R2) fails since $1 \vee (1 \vee -1) = 1 \vee 0 = 1$, but $(1 \vee 1) \vee -1 = 1 \vee -1 = 0$

and it is equally easy to check the other identities hold

$A_3 = \{0, 1\}$ with $\vee =$ join, $-x = x^\smile = x$, $; = \wedge$, and $1 = 1$

Fact 1: For a group G the **complex algebra** $G^+ = (\mathcal{P}(G), \cup, -, ;, ^\smile, \{e\})$ is a (representable) relation algebra where $X; Y = \{xy : x \in X, y \in Y\}$ and $X^\smile = \{x^{-1} : x \in X\}$

For $b \in A$, define the **relativization** $A{\restriction}b = (\{a \wedge b : a \in A\}, \vee, -^b, ;^b, ^\smile, 1)$

where $1 \leq b = b^\smile$, $-^b x = -x \wedge b$ and $x ;^b y = x; y \wedge b$

Fact 2: $A{\restriction}b$ satisfies (R1-3,5-10) and (R4) $\iff$ $b; b \leq b$

$A_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2)^+ \restriction \{(0, 0), (0, 1), (1, 0)\}$ has 8 elements

$A_5 =$ 2-element Boolean algebra with $x;y = 0$, $x^{\smile} = x$ and $1 = 1$

$A_6 =$ 2-element Boolean algebra with $x;y = x \wedge y$, $x^{\smile} = 0$ and $1 = 1$

$A_7 = \{\bot, 1, -1, \top\}$ a BA with $x;y = \begin{cases} x & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$, $x^{\smile} = x$ and $1 = 1$

$A_8 = \{0, 1\}$ a BA with $x;y = \begin{cases} 1 & \text{if } x, y = 0 \\ x \wedge y & \text{otherwise} \end{cases}$, $x^{\smile} = x$ and $1 = 1$

$A_9 = \mathbb{Z}_3^+$, but for $x \in \{1, 2\}$ and $y, z \in \mathbb{Z}_3$ redefine $\{x\}^{\smile} = \{x\}$ and $\{x\};\{y, z\} = \{x^{-1} \cdot y, x^{-1} \cdot z\}$ where $^{-1}$, $\cdot$ are the group operations in $\mathbb{Z}_3$

$A_{10} = \{0, 1\}$ a BA with $; = \vee$, $x^{\smile} = x$, $1_{RA} = 0$

In each case one needs to check that $A_i \not\models$ (Ri), but the other axioms hold:

(R10) let $x{=}y{=}1$ in $x^{\smile}; -(x;y) \vee -y = 1 \vee -(1 \vee 1) \vee -1 = 1 \neq 0 = -y$

A variant of Tarski's axioms

Theorem (Andreka, Givant, J., Nemeti)

The identities (R1)-(R10) *are an independent basis for RA.*

Somewhat surprisingly, it turns out that by modifying (R8) slightly, (R7) becomes redundant:

Let $\mathcal{R} =$ (R1)-(R6),(R9),(R10) plus (R8') $= x;(y \vee z) = x;y \vee x;z$

(R1) $x \vee y = y \vee x$
(R2) $x \vee (y \vee z) = (x \vee y) \vee z$
(R3) $\neg(\neg x \vee y) \vee \neg(\neg x \vee \neg y) = x$
(R4) $x;(y;z) = (x;y);z$
(R5) $x;1 = x$
(R6) $x^{\smile\smile} = x$
(R7) $(x;y)^{\smile} = y^{\smile};x^{\smile}$
(R8') $x;(y \vee z) = x;y \vee x;z$
(R9) $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$
(R10) $x^{\smile};\neg(x;y) \vee \neg y = \neg y$

Theorem (Andreka, Givant, J., Nemeti)

The identities $\mathcal{R}$ *are also an independent basis for RA.*

Another variant of Tarski's axioms

Let $\mathcal{S}$ = (R1)-(R6),(R8),(R8'),(R10)

(R1) $x \vee y = y \vee x$
(R2) $x \vee (y \vee z) = (x \vee y) \vee z$
(R3) $\neg(\neg x \vee y) \vee \neg(\neg x \vee \neg y) = x$
(R4) $x;(y;z) = (x;y);z$
(R5) $x;1 = x$

(R6) $x^{\smile\smile} = x$
(R7) $(x;y)^{\smile} = y^{\smile};x^{\smile}$
(R8) $(x \vee y);z = x;z \vee y;z$
(R8') $x;(y \vee z) = x;y \vee x;z$
(R9) $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$
(R10) $x^{\smile};\neg(x;y) \vee \neg y = \neg y$

Theorem (Andreka, Givant, J., Nemeti)

The identities $\mathcal{S}$ are also an independent basis for RA.

The independence models $A_1 - A_{10}$ are modified somewhat for these proofs.

All models are minimal in size and the paper also describes other models.

Nonclassical axiomatization of relation algebras

An **idempotent semiring** (ISR) is of the form $(A, \vee, \cdot, 1)$ where

- $(A, \vee)$ is a **semilattice** (i.e., $\vee$ is assoc., comm., idempotent)
- $(A, \cdot, 1)$ is a **monoid**
- $x(y \vee z) = xy \vee xz$ and $(x \vee y)z = xz \vee yz$

Residuated lattices (RL) are ISRs expanded with $\wedge, \backslash, /$

Involutive residuated lattices (InRL) are RLs expanded with $0, \sim, -$ such that $\sim x = x \backslash 0$, $-x = 0/x$ and $-\sim x = x = \sim - x$

Cyclic residuated lattices are InRLs that satisfy $\sim x = -x$

Generalized bunched implication algebras are RLs expanded with $\rightarrow$

Residuated monoids (RM) are Boolean residuated lattices

Relations algebras are RMs with $x^{\smile} = \neg{\sim} x$, $(xy)^{\smile} = y^{\smile} x^{\smile}$

A short biography of Bjarni Jónsson

Born on February 15, 1920 in **Draghals, Iceland**

B. Sc. from UC Berkeley in 1943

Ph. D. from UC Berkeley in 1946 under **Alfred Tarski**

1946-1956 **Brown University**

1956-1966 **University of Minnesota**

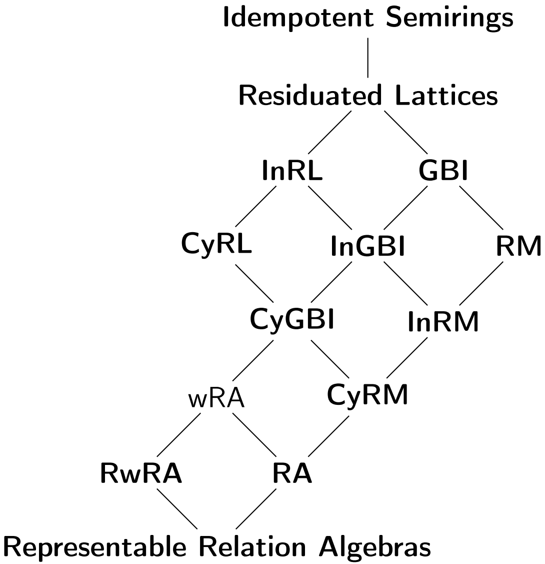
1966-1993 **Vanderbilt University, first distinguished professor**

1974 invited speaker at **International Congress of Mathematicians**

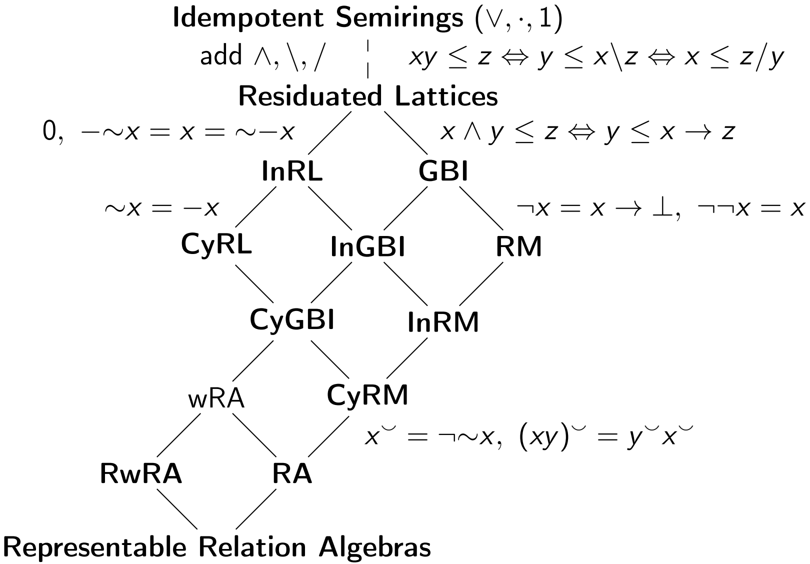
2012 elected inaugural fellow of the **American Mathematical Society**

13 Ph. D. students, 73 Ph. D. descendants

Varieties of partially ordered algebras



Varieties of partially ordered algebras



Residuated lattices

A ***residuated lattice*** is of the form $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \backslash, /)$ where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and $\backslash, /$ are the **left** and **right residuals** of $\cdot$, i.e., for all $x, y, z \in A$

$$xy \leq z \iff y \leq x\backslash z \iff x \leq z/y.$$

The previous formula is equivalent to the following 4 identities:

$$x \leq y\backslash(yx \vee z) \qquad x((x\backslash y) \wedge z) \leq y$$

$$x \leq (xy \vee z)/y \qquad ((x/y) \wedge z)y \leq x$$

so residuated lattices form a variety.

For an **arbitrary constant** 0 in a residuated lattice define the ***linear negations*** $\sim x = x\backslash 0$ and $-x = 0/x$

An ***involutive residuated lattice*** is a residuated lattice s.t. $\sim -x = x = -\sim x$

Involutive residuated lattices

Alternatively, $(A, \wedge, \vee, \cdot, 1, 0, \sim, -)$ is an **involutive residuated lattice** if $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, $\sim -x = x = -\sim x$, $0 = -1$ and $x \leq -y \iff xy \leq 0$.

It follows that $x \backslash y = \sim(-y \cdot x)$ and $x/y = -(y \cdot \sim x)$.

An involutive residuated lattice is ***cyclic*** if $\sim x = -x$

E.g. a **relation algebra** $(A, \wedge, \vee, \neg, \cdot, ^{\smile}, 1)$ is a **cyclic involutive residuated lattice** if one defines $x \backslash y = \neg(x^{\smile} \cdot \neg y)$, $x/y = \neg(\neg x \cdot y^{\smile})$ and $0 = \neg 1$, and omits the operations $\neg, ^{\smile}$ from the signature

The **cyclic linear negation** is given by $\sim x = \neg(x^{\smile}) = (\neg x)^{\smile}$

The variety of **(cyclic) involutive residuated lattices** has a **decidable equational theory** while this is not the case for relation algebras

Generalized bunched implication algebras

A **generalized bunched implication algebra** $(A, \wedge, \vee, \to, \top, \bot, \cdot, 1, \backslash, /)$ is a **residuated lattice** $(A, \wedge, \vee, \cdot, 1, \backslash, /)$ such that $(A, \wedge, \vee, \to, \top, \bot)$ is a **Heyting algebra**, i.e., $\top, \bot$ are top and bottom elements and

$$x \wedge y \leq z \iff y \leq x \to z$$

or equivalently the following 2 identities hold

$$x \leq y \to ((x \wedge y) \vee z) \qquad x \wedge (x \to y) \leq y$$

Theorem (Galatos and J.)

*The variety **GBI** of generalized bunched implication algebras has the finite model property, hence a decidable equational theory*

The intuitionistic negation is defined as $\neg x = x \to \bot$

RA $=$ cyclic involutive **GBI** $\cap$ $\mathrm{Mod}(\neg\neg x = x, \neg{\sim}(xy) = (\neg{\sim}y)(\neg{\sim}x))$

Number of nonisomorphic algebras

Number of elements: $n =$	1	2	3	4	5	6	7	8
Residuated lattices	1	1	3	20	149	1488	18554	295292
GBI-algebras	1	1	3	20	115	899	7782	80468
Bunched impl. algebras	1	1	3	16	70	399	2261	14358
Involutive resid. lattices	1	1	2	9	21	101	284	1464
Cyclic inv. resid. lattices	1	1	2	9	21	101	279	1433
Invol. GBI-algebras	1	1	2	9	8	43	49	282
Cyclic inv. GBI-algebras	1	1	2	9	8	43	48	281
Invol. BI-algebras	1	1	2	9	8	42	46	263
Res. Bool. Monoids (RM)	1	1	0	5	0	0	0	25
Classical relation algebras	1	1	0	3	0	0	0	13

Weakening relations

Recall that RA and RRA both have **undecidable** equational theories

I. Nemeti [1987] proved that **removing associativity** from the basis of RA the resulting variety NRA of **nonassociative relation algebras** has a **decidable** equational theory

N. Galatos and J. [2012] showed that if **classical negation** in RA is weakened to a **De Morgan negation** then the resulting variety qRA of **quasi relation algebras** has a **decidable** equational theory

However there are **no natural models using binary relations**

Let $\mathbf{P} = (P, \sqsubseteq)$ be a **partially ordered** set

Let $Q \subseteq P^2$ be an **equivalence relation** that contains $\sqsubseteq$, and define the set of ***weakening relations*** on $\mathbf{P}$ by $\mathsf{Wk}(\mathbf{P}, Q) = \{\sqsubseteq \circ R \circ \sqsubseteq \,:\, R \subseteq Q\}$

Since $\sqsubseteq$ is **transitive and reflexive** $\mathsf{Wk}(\mathbf{P}, Q) = \{R \subseteq Q : \sqsubseteq \circ R \circ \sqsubseteq = R\}$

Full weakening relation algebras

If $Q = P \times P$ write $\mathrm{Wk}(\mathbf{P})$ and call it the ***full weakening relation algebra***

Weakening relations are the **analogue of binary relations** when the category **Set** of sets and functions is replaced by the category **Pos** of partially ordered sets and order-preserving functions

Since sets can be considered as **discrete posets** (i.e. ordered by the identity relation), **Pos** contains **Set** as a full subcategory, which implies that weakening relations are a **substantial generalization** of binary relations

However, weakening relations do not allow $\neg$ or $^{\smile}$ as operations

They have applications in **sequent calculi**, **proximity lattices/spaces**, **order-enriched categories**, **cartesian bicategories**, **bi-intuitionistic modal logic**, **mathematical morphology** and **program semantics**, e.g. via **separation logic**

A small example

Let $C_2 = \{0, 1\}$ be the two element chain with $0 \sqsubseteq 1$

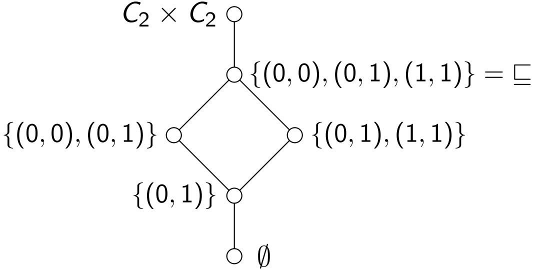


Figure: The full weakening relation algebra $\mathbf{Wk}(\mathbf{C}_2)$

Operations on weakening relations

$\mathrm{Wk}(\mathbf{P}, Q)$ is a **complete** and **perfect** distributive lattice under $\cup, \cap$

$\implies$ can expand $\mathrm{Wk}(\mathbf{P}, Q)$ to a **Heyting algebra** by adding $\to$

Weakening relations are closed under **composition**: for $R, S \in \mathrm{Wk}(\mathbf{P}, Q)$

$R \circ S = (\sqsubseteq \circ R \circ \sqsubseteq) \circ (\sqsubseteq \circ S \circ \sqsubseteq) = \sqsubseteq \circ (R \circ \sqsubseteq \circ S) \circ \sqsubseteq \in \mathrm{Wk}(\mathbf{P}, Q)$

$\sqsubseteq$ is an **identity element** for composition: $R \circ \sqsubseteq = R = \sqsubseteq \circ R$

$\circ$ distributes over arbitrary unions, so we can add **residuals** $\backslash, /$

So $\mathrm{Wk}(\mathbf{P}, Q)$ is a **distributive residuated lattice** with Heyting implication

Representable intuitionistic relation algebras

Theorem

*$\mathbf{Wk}(\mathbf{P}, Q) = (\mathrm{Wk}(\mathbf{P}, Q), \cap, \cup, \to, Q, \emptyset, \circ, \sqsubseteq, \sim)$ is a **cyclic involutive GBI-algebra**.*

In particular, $\top = Q$, $\quad \bot = \emptyset$,

$R \to S = (\sqsupseteq \circ (R \cap S') \circ \sqsupseteq)'$ where $S' = Q - S$,

and $\sim R = R^{\smile\prime} = R^{\prime\smile}$.

In $\mathbf{Wk}(\mathbf{P})$, if $R \neq \bot$ then $\top \circ R \circ \top = \top$

If $\mathbf{P}$ is a discrete poset then $\mathbf{Wk}(\mathbf{P}) = \mathbf{Rel}(P)$ is the full representable relation algebra on the set P

So algebras of weakening relations are like representable relation algebras

Define the class RwRA of ***representable weakening relation algebras*** as all algebras that are **embedded** in a weakening algebra $\mathbf{Wk}(\mathbf{P}, Q)$ for some poset $\mathbf{P}$ and equivalence relation Q that contains $\sqsubseteq$

In fact the variety RRA is a finitely axiomatizable subvariety of RwRA

Theorem

1. RwRA *is a* ***discriminator variety*** *with* $x \leftrightarrow y = (x \to y) \wedge (y \to x)$ $t(x, y, z) = (x \wedge \top(\sim(x \leftrightarrow y)\top) \vee (z \wedge \sim\top(\sim(x \leftrightarrow y)\top)$
2. RRA *is the subvariety of* RwRA *defined by* $\neg\neg x = x$
3. RwRA *is not finitely axiomatizable relative to the variety* GBI

Poset semantics of weakening relations

Birkhoff showed that a finite distributive lattice $\mathbf{A}$ is determined by its poset $J(\mathbf{A})$ of completely join-irreducible elements (with the order induced by $\mathbf{A}$)

The result also holds for complete perfect distributive lattices

Conversely, if $\mathbf{Q} = (Q, \leq)$ is a poset, then the set of **downward-closed** subsets $D(\mathbf{Q})$ of $\mathbf{Q}$ forms a complete perfect distributive lattice under intersection and union

$D(\mathbf{Q})$ is a Heyting algebra, with $U \to V = Q - {\uparrow}(U - V)$ for any $U, V \in D(\mathbf{Q})$

For a poset $\mathbf{P}$, $\mathrm{Wk}(\mathbf{P})$ is complete and perfect and $J(\mathrm{Wk}(\mathbf{P})) \cong \mathbf{P} \times \mathbf{P}^{\partial}$

The composition $\circ$ of $\mathbf{Wk}(\mathbf{P})$ is determined by its restriction to pairs of $\mathbf{P} \times \mathbf{P}^{\partial}$, where $\circ$ is a partial operation given by

$$(t, u) \circ (v, w) = \begin{cases} (t, w) & \text{if } u = v \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Semantics of relation algebras

For comparison, we first consider the case of relation algebras.

A complete perfect relation algebra has a complete atomic Boolean algebra as reduct, and the set of join-irreducibles is the set of atoms.

B. Jónsson and A. Tarski [1952] showed the operation of composition, restricted to atoms, is a partial operation precisely when the atoms form a **(Brandt) groupoid**, or equivalently a **small category with all morphism being invertible**.

For Heyting relation algebras we have a similar result using **partially-ordered groupoids**

Groupoids and partially ordered groupoids

A ***groupoid*** is defined as a **partial** algebra $\mathbf{G} = (G, \circ, ^{-1})$ such that $\circ$ is a **partial** binary operation and $^{-1}$ is a (total) unary operation on G that satisfy:

1. $(x \circ y) \circ z \in G$ or $x \circ (y \circ z) \in G \implies (x \circ y) \circ z = x \circ (y \circ z)$,
2. $x \circ y \in G \iff x^{-1} \circ x = y \circ y^{-1}$,
3. $x \circ x^{-1} \circ x = x$ and $x^{-1-1} = x$.

These axioms imply $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$

Typical examples of groupoids are **disjoint unions of groups** and the ***pair-groupoid*** $(X \times X, \circ, ^{\smile})$

Partially ordered groupoid semantics

A ***partially-ordered groupoid*** $(G, \leq, \circ, ^{-1})$ is a groupoid $(G, \circ, ^{-1})$ such that $(G, \leq)$ is a poset and $\circ, ^{-1}$ are order-preserving:

- $x \leq y$ and $x \circ z, y \circ z \in G \implies x \circ z \leq y \circ z$
- $x \leq y \implies y^{-1} \leq x^{-1}$
- $x \leq y \circ y^{-1} \implies x \leq x \circ x^{-1}$

If $\mathbf{P} = (P, \leq)$ a poset then $\mathbf{P} \times \mathbf{P}^{\partial} = (P \times P, \leq, \circ, ^{\smile})$ is a partially-ordered groupoid with $(a, b) \leq (c, d) \iff a \leq c$ and $d \leq b$.

Theorem

Let $\mathbf{G} = (G, \leq, \circ, ^{-1})$ be a partially-ordered groupoid. Then $D(\mathbf{G})$ is a cyclic involutive GBI-algebra.

Semantics for full weakening relation algebras

In fact for a poset $\mathbf{P} = (P, \sqsubseteq)$ the weakening relation algebra $\mathbf{Wk}(\mathbf{P})$ is obtained from the partially-ordered pair-groupoid $\mathbf{G} = \mathbf{P} \times \mathbf{P}^\partial$

For example, the 3-element chain $\mathbf{C}_3$ gives a 9-element partially ordered groupoid, and $\mathbf{Wk}(\mathbf{C}_3)$ has 20 elements

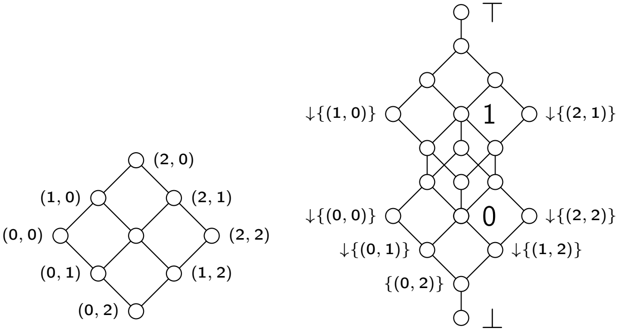


Figure: The weakening relation $\mathbf{Wk}(\mathbf{C}_3)$ and its po-pair-groupoid

Cardinality of $\mathbf{Wk}(\mathbf{C}_n)$

Theorem

For an n-element chain $\mathbf{C}_n$ the weakening relation algebra $\mathbf{Wk}(\mathbf{C}_n)$ has cardinality $\binom{2n}{n}$.

Proof.

This follows from the observation that $D(\mathbf{C}_m \times \mathbf{C}_n)$ has cardinality $\binom{m+n}{n}$
For $n = 1$ this holds since an m-element chain has $m + 1$ down-closed sets
Assuming the result holds for n, note that $\mathbf{P} = \mathbf{C}_m \times \mathbf{C}_{n+1}$ is the disjoint union of $\mathbf{C}_m \times \mathbf{C}_n$ and $\mathbf{C}_m$, where we assume the additional m elements are not below any of the elements of $\mathbf{C}_m \times \mathbf{C}_n$
The number of downsets of $\mathbf{P}$ that contain an element a from the extra chain $\mathbf{C}_m$ as a maximal element is given by $\binom{k+n}{n}$ where k is the number of elements above a
Hence the total number of downsets of $\mathbf{P}$ is $\sum_{k=0}^{m} \binom{k+n}{n} = \binom{m+n+1}{n+1}$. $\square$

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Thank You, Bjarni ! (July 1990)