Partially ordered varieties of idempotent residuated posets

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based on joint work with
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TACL, University of Nice
June 17 - 21, 2019

Outline

- Involutive residuated posets
- Partially ordered subvarieties
- Number of finite members in some subvarieties
- Boolean decomposition of commutative idempotent involutive residuated posets
- Cyclic idempotent involutive residuated posets are commutative

Involutive residuated posets

An **involutive residuated poset** is of the form $(A, \leqslant, \cdot, \sim, -, 0)$ such that

1. $(A, \leqslant)$ is a **poset** (i.e., $\leqslant$ is reflexive, antisymmetric, transitive),
2. $\cdot$ is an **associative** operation on A: $(xy)z = x(yz)$, and
3. $x \leqslant y \iff x \cdot {\sim}y \leqslant 0 \iff -y \cdot x \leqslant 0$ for all $x, y \in A$.

The element -0 is denoted by 1, and $x \cdot y$ is usually written xy.

Also define $x + y = {\sim}(-y \cdot -x)$ (not necessarily commutative).

Involutive residuated posets

Lemma

Involutive residuated posets have the following properties:

1. $\sim - x = x = - \sim x$
2. $x \leqslant y \iff \sim y \leqslant \sim x \iff -y \leqslant -x$
3. $1x = x = x1$
4. $1 = \sim 0, \quad -1 = \sim 1 = 0$
5. $\sim(-y \cdot -x) = -(\sim y \cdot \sim x)$
6. $xy \leqslant z \iff y \leqslant \sim(-z \cdot x) \iff x \leqslant -(y \cdot \sim z)$

Hence they are ***residuated po-monoids*** *with* ***residuals*** $x \backslash y = \sim(-y \cdot x)$ *and* $x/y = -(y \cdot \sim x)$, *and* $\cdot$ *is* ***order-preserving*** *in both arguments.*

Involutive residuated posets

Involutive "residuation": $x \leqslant y \iff x \cdot {\sim} y \leqslant 0 \iff -y \cdot x \leqslant 0$

Proof of (1): ${\sim}{-}x = x = -{\sim}x$.

$-x \leqslant y \iff -x \cdot {\sim} y \leqslant 0 \iff {\sim} y \leqslant x$ (dual Galois connection).

Therefore $-x \leqslant -x \implies {\sim}{-}x \leqslant x$, hence ${\sim}{-}{\sim}{-}x \leqslant {\sim}{-}x \leqslant x$.

Equivalently $-x \leqslant -{\sim}{-}x$.

Similarly ${\sim}{-}x \leqslant {\sim}{-}x \implies -{\sim}{-}x \leqslant -x$, hence $-{\sim}{-}x = -x$.

Now $x \leqslant x \implies -x \cdot x \leqslant 0$, so $-{\sim}{-}x \cdot x \leqslant 0$ and therefore $x \leqslant {\sim}{-}x$.

This proves ${\sim}{-}x = x$, and $-{\sim}x = x$ follows similarly. □

(2)-(6) are also easy to derive.

Involutive residuated posets are a po-variety

The class of involutive residuated posets is denoted by **InRP**.

All operations are order-preserving or order-reversing in each argument, hence this class forms a **partially ordered quasivariety** (Pigozzi 2004)

InRP is a **partially ordered variety** (or po-variety) defined by the po-identities

$$(xy)z = x(yz), \quad \sim\!-x = x = -\!\sim\!x, \quad \sim\!0 = -0$$

$$-0 \cdot x = x, \quad -x \cdot x \leqslant 0, \quad x \cdot \sim\!(yx) \leqslant \sim\!y$$

together with the order-preservation of $\cdot$ and the order-reversal of $\sim, -$.

Integral, cyclic, commutative and idempotent InRPs

IInRP is the po-subvariety of **integral** ($x \leqslant 1$) InRPs

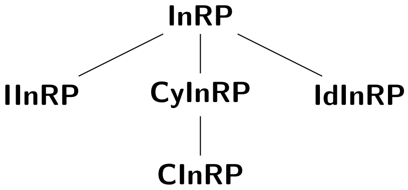
CyInRP is the po-subvariety of **cyclic** ($\sim x = -x$) InRPs

CInRP is the po-subvariety of **commutative** ($xy = yx$) InRPs

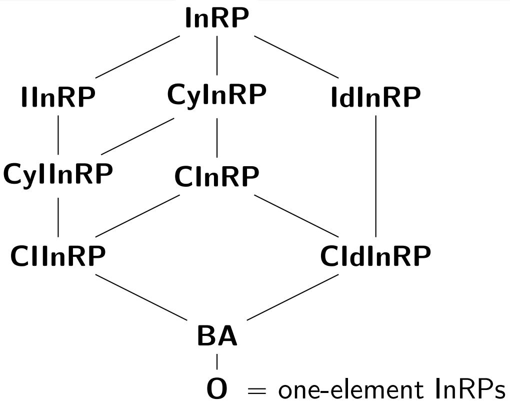
IdInRP is the po-subvariety of **idempotent** ($xx = x$) InRPs

Commutative $\implies$ cyclic:

$x \leqslant {\sim} y \iff -{\sim} y \cdot x \leqslant 0 \iff x \cdot {\sim}{-}y \leqslant 0 \iff x \leqslant -y.$



Po-subvarieties of involutive residuated posets



Note: meet in the diagram = intersection (joins are not shown)

Integral + idempotent $\implies$ Boolean

Cyclic + idempotent $\implies$ commutative (more details and/or proofs later)

Po-subvarieties of involutive residuated posets

InRP contains several well-known subclasses of (po-)algebras:

- The variety of **pointed groups** is axiomatized by adding $x \leqslant y \implies x = y$ to InRP.
- The variety of **groups** is axiomatized by adding $0 = 1$ to pointed groups. Hence involutive residuated posets may be considered the analogue of (pointed) groups over the category of posets.
- The po-subvariety of **pregroups** (Lambek 1999) is obtained by adding the identity $xy = {\sim}(-y \cdot -x)$ to **InRP**.
- The po-subvariety of **partially ordered groups** (Fuchs 1963, Glass 1999) is obtained by adding ${\sim}x = -x$ to pregroups.

Po-subvarieties of involutive residuated posets

- **Involutive pocrims** (Raftery 2007) are defined as **commutative integral involutive residuated partially ordered monoids**, hence they are the same as **CIInRP**.

  They are a class of **algebras** since $x \leqslant y \iff -y \cdot x = 0$.

  Involutive pocrims include the subvarieties of **IMTL-algebras**, **MV-algebras** and **Boolean algebras**.

- The variety of **involutive residuated lattices** is the expansion of InRP with a semilattice operation $\vee$ such that $x \leqslant y \iff x \vee y = y$.

  This class includes the subvarieties of **lattice-ordered groups**, **classical linear logic algebras** (without exponentials), **De Morgan monoids** and **Sugihara algebras** from relevance logic.

Number of nonisomorphic po-algebras

Number of elements: $n =$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Residuated posets	1	2	5	28	186	1795		
Residuated lattices	1	1	3	20	149	1488	18554	295292

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Comm. residuated posets	1	2	5	24	131	1001		
Comm. residuated lattices	1	1	3	16	100	794	7493	84961

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Idempotent inv. res. posets	**1**	**1**	**1**	**2**	**2**	**4**	**4**	**9**
MV-algebras	1	1	1	2	1	2	1	3

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Idempotent inv. res. posets	**1**	**1**	**1**	**2**	**2**	**4**	**4**	**9**
MV-algebras	1	1	1	2	1	2	1	3
Boolean algebras	1	1	0	1	0	0	0	1

Number of nonisomorphic **idempotent** InRPs

There are "very few" **idempotent** involutive residuated posets

$n =$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
IdInRP	1	1	1	2	2	4	4	9	10	22	24	53	61	134	157	343
IdInRL	1	1	1	2	2	4	4	9	10	21	22	49	52	114	121	270
MV	1	1	1	2	1	2	1	3	2	2	1	4	1	2	2	5
BA	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1

Most of the idempotent InRPs are **lattice-ordered**.

All with $\leqslant 16$ elements are **commutative**!

For an idempotent InRP define the **monoid preorder** by $x \sqsubseteq y \iff xy = x$.

1 is the **top** of this preorder; if $\bot$ exists then $\bot \sqsubseteq x$

Note: If $\cdot$ is **commutative** then $\sqsubseteq$ is a (meet-)**semilattice** order.

Clearly the semilattice order $\sqsubseteq$ determines the moniod operation $\cdot$

A typical finite commutative idempotent involutive RL

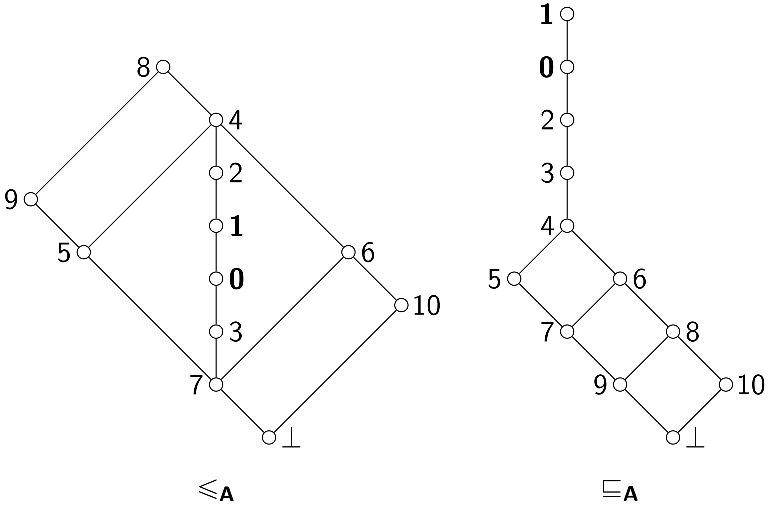
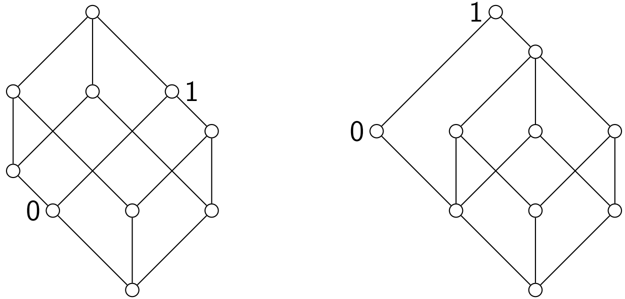


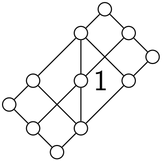
Figure: The lattice order and the monoid order for $\mathbf{A} \in \mathsf{IdInRL}$

The **smallest** commutative idempotent invol. res. **poset**

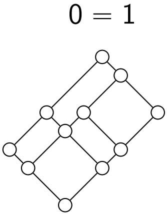


$\leqslant_{10}$ $\sqsubseteq_{10}$

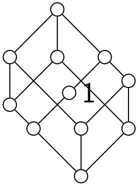
The next two **smallest** idempotent invol. res. **posets**



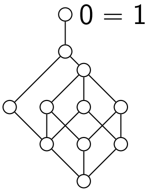
$\leqslant_{11,1}$



$\sqsubseteq_{11,1}$



$\leqslant_{11,2}$



$\sqsubseteq_{11,2} = \sqsubseteq_{10} \oplus \mathbf{1}$

Constants in cyclic idempotent involutive posets

A residuated lattice is said to be **square-increasing** if it satisfies the identity $x \leqslant x^2$, and *square-decreasing* if $x^2 \leqslant x$.

Lemma

Given a ***square-increasing*** *involutive residuated poset* $\mathbf{A}$*,*

$$0 \leqslant 1 \iff \mathbf{A} \text{ is idempotent.}$$

Proof.

It suffices to show that in a square-increasing involutive residuated poset, $0 \leqslant 1 \iff xx \leqslant x$.
If $\mathbf{A}$ is square-decreasing then $00 \leqslant 0$, and then $0 \leqslant 0\backslash 0 = 1$.
Conversely, suppose that $0 \leqslant 1$. Then $-x = 0/x \leqslant 1/x$, hence $-xx \leqslant 1$.
By square-increasing, $-xx \leqslant (-x-x)x = -x(-xx) \leqslant -x1 = -x$.
Hence, $x \leqslant \sim(-xx) = x\backslash x$, and therefore $x^2 \leqslant x$. □

Boolean intervals in commutative IdInRLs

Corollary

*In any **idempotent** involutive residuated poset* $0 \leqslant 1$.

In an involutive residuated lattice, idempotence implies that $0 \leqslant 1$ and that $([0,1], \cdot, +, -, 0, 1)$ is a Boolean algebra, where $x + y = \sim(-y \cdot -x)$.

For $A \in$ **CIdInRP**, define the **terms**

$0_x = x \cdot -x$ and

$1_x = -0_x = -(x \cdot -x) = x + -x$.

Define the **monoid interval of x** by $\mathbb{B}_x = \{a \in A : 0_x \sqsubseteq a \sqsubseteq 1_x\}$

I.e., $\mathbb{B}_x = \{a \in A : 0_x \cdot a = 0_x \text{ and } a \cdot 1_x = a\}$

Intervals in the monoid order of CIdInRPs

Lemma

For $a, b \in [0,1]$, $a \sqsubseteq b \iff a \leqslant b$, *hence* $\mathbb{B}_0 = \mathbb{B}_1 = [0,1]$.

Lemma (PJ., Olim Tuyt, Diego Valota)

Let $A \in$ **CIdInRP**, $x \in A$ *and* $a \in \mathbb{B}_x$. *Then*

1. $-a = a \to 0_x$
2. $-a \in \mathbb{B}_x$, *and*
3. $a \cdot -a = 0_x$.

Theorem (PJ., Olim Tuyt, Diego Valota)

Let A *be a* ***commutative idempotent involutive residuated poset****.*
Then for all $x \in A$, $(\mathbb{B}_x, \cdot, +, -, 0_x, 1_x)$ *is a Boolean algebra.*

Boolean intervals partition any CIdInRP

The set of monoid intervals $\mathbb{B}_x$ actually partition A.

To see this, define a relation $\equiv_0$ as follows for $x, y \in A$

$$x \equiv_0 y \qquad \Longleftrightarrow \qquad 0_x = 0_y.$$

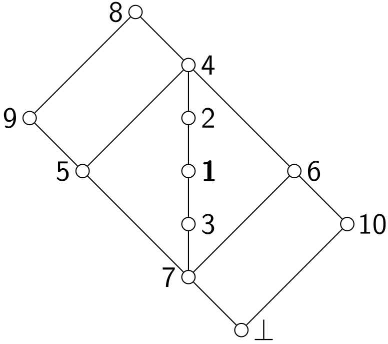
$\equiv_0$ is easily seen to be an **equivalence relation** on A.

Let $[x]_0$ denote the equivalence class of an element $x \in A$.

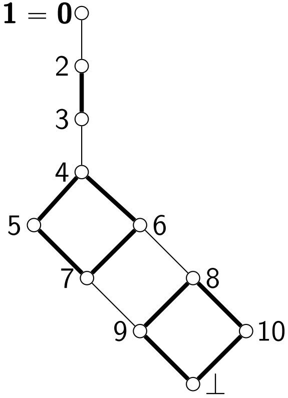
Theorem (PJ., Olim Tuyt, Diego Valota)

For all $x \in A$, $[x]_0 = \mathbb{B}_x$.

A typical finite commutative idempotent involutive RL

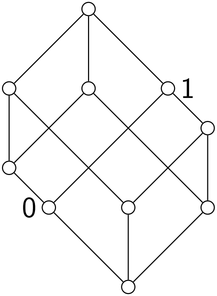


$\leqslant_{\mathbf{A}}$

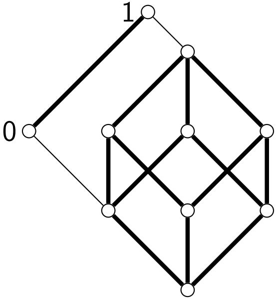


$\sqsubseteq_{\mathbf{A}}$

The **smallest** commutative idempotent invol. res. **poset**



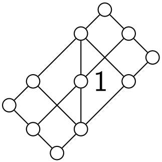
$\leqslant_{10}$



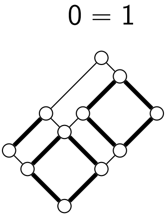
$\sqsubseteq_{10}$

Dark lines show the monoid order **partitioned** into **Boolean intervals**

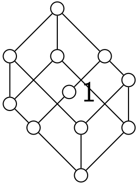
The next two **smallest** idempotent invol. res. **posets**



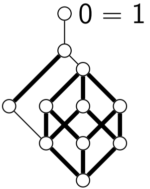
$\leqslant_{11,1}$



$\sqsubseteq_{11,1}$



$\leqslant_{11,2}$



$\sqsubseteq_{11,2} = \sqsubseteq_{10} \oplus \mathbf{1}$

Dark lines show the monoid order **partitioned** into **Boolean intervals**

Properties of cyclic idempotent involutive posets

Idempotence for cyclic involutive residuated posets is a **strong restriction**.

Lemma (José Gil-Ferez and PJ)

Any involutive idempotent residuated posets satisfies:

1. $x({\sim}x) \leqslant {\sim}x$ *and* $(-x)x \leqslant -x$,
2. $x({\sim}x) \leqslant x$ *and* $(-x)x \leqslant x$.

Assuming ***cyclicity*** *implies the following additional identities:*

3. $x({\sim}x)x = x({\sim}x)$,
4. $x({\sim}x) = ({\sim}x)x$.

Proof.

In any involutive residuated poset $\sim(yx) \leqslant \sim(yx)$, so $yx(\sim(yx)) \leqslant 0$, whence $x(\sim(yx)) \leqslant \sim y$.

1. Follows from this identity and idempotence by substituting x for y.
2. Replace x by $\sim x$ in the second identity of (1).
3. Multiplying (1) by x on the right we obtain $x(\sim x)x \leqslant (\sim x)x$. By cyclicity $(\sim x)x \leqslant 0$, and using idempotence gives $xx(\sim x)x \leqslant 0$, or equivalently $x(\sim x)x \leqslant \sim x$. Multiplying by x on the left shows that $x(\sim x)x \leqslant x(\sim x)$. Multiplying (2) by $x(\sim x)$ on the left produces $x(\sim x)x(\sim x) \leqslant x(\sim x)x$, whence $x(\sim x) \leqslant x(\sim x)x$ follows from idempotence. Therefore (3) holds.
4. Again multiplying (1) by x on the right we obtain $x(\sim x)x \leqslant (\sim x)x$, hence by (3) we get $x(\sim x) \leqslant (\sim x)x$. Using cyclicity we can replace x by $\sim x$ to deduce the reverse inequality.

□

Every cyclic idempotent involutive poset is commutative

Theorem (José Gil-Ferez and PJ)

Every ***cyclic idempotent*** *involutive residuated poset is* ***commutative****.*

Proof.

The identity $y \cdot {\sim}(xy) \leqslant {\sim}x$ holds in any InRL, hence

$$xy \cdot {\sim}(xy) \leqslant x \cdot {\sim}x \leqslant {\sim}x.$$

Applying (4) of the preceeding lemma on the left, we have ${\sim}(xy)xy \leqslant {\sim}x$, from which we deduce ${\sim}(xy)xyx \leqslant ({\sim}x)x \leqslant 0$. Therefore $xyx \leqslant xy$.

Now multiply both sides by y on the left and use idempotence to deduce the identity $yx \leqslant yxy$. Renaming variables proves $xyx = xy$.

A similar argument shows $xyx = yx$, whence $xy = xyx = yx$. □

A noncyclic idempotent involutive residuated lattice

There exist **noncommutative** idempotent involutive residuated lattices:

Example (Jóse Gil-Ferez and PJ)

Let $A = \mathbb{Z} \oplus \{\mathbf{1}\} \oplus \mathbb{Z}^\partial$, where $\oplus$ is the ordinal sum.

Lattice order:

$\cdots a_{-2} < a_{-1} < a_0 < a_1 < a_2 \cdots < \mathbf{1} < \cdots b_2 < b_1 < b_0 < b_{-1} < b_{-2} \cdots$

Monoid preorder:

$\cdots a_{-2} \equiv b_{-2} \sqsubset a_{-1} \equiv b_{-1} \sqsubset a_0 \equiv b_0 \sqsubset a_1 \equiv b_1 \sqsubset a_2 \equiv b_2 \sqsubset \cdots \sqsubset \mathbf{1}$

Linear negations:

$\mathbf{1} = \mathbf{0}$, $\;{\sim}a_i = b_i$, $\;{\sim}b_i = a_{i-1}$, $\;-a_i = b_{i+1}$, $\;-b_i = a_i$

Hence ${\sim}{\sim}a_i = a_{i-1}$ and $--a_i = a_{i+1}$ and the same for b_i.

Conjecture: All **finite** idempotent involutive res. posets are **commutative**.

Some partial results

Theorem

Finite idempotent involutive residuated chains are commutative.

The following results have been obtained using Prover9 [McCune]

Theorem

1. *The po-subvariety of IdInRP determined by the identity* $----x = x$ *satisfies* $--x = x$*, hence is cyclic and thus commutative.*
2. *The po-subvariety of IdInRP determined by the identity* $-------x = x$ *satisfies* $----x = x$.

Let $-_n x$ be the term with n copies of $-$. Then $-_n x$ is a permutation on A, hence if A is **finite** it satisfies $-_n x = -_m x$ for some $n > m \geqslant 0$. Applying m copies of $\sim$ on both sides shows A satisfies $-_{n-m} x = x$.

Some references

A. M. W. Glass: "Partially ordered groups." World Scientific, 1999.

J. Lambek: *Type grammar revisited*, In A. Lecomte, F. Lamarche and G. Perrier, editors, Logical Aspects of Computational Linguistics, Springer LNAI 1582, 1999, 1–27.

W. McCune: *Prover9 and Mace4*. `www.cs.unm.edu/~mccune/prover9/`, 2005–2010.

D. Pigozzi: *Partially ordered varieties and quasivarieties*. Unpublished lecture notes, 2004, 1–26, `orion.math.iastate.edu/dpigozzi/notes/santiago_notes.pdf`.

J. Raftery: *On the variety generated by involutive pocrims*. Reports on Mathematical Logic, 42, (2007), 71–86.

Thanks!