A full description of finite commutative idempotent involutive residuated lattices

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Overview

- Definition of commutative idempotent involutive residuated lattices
- Examples
- Some properties
- Gluing construction
- Ungluing decomposition
- Applications
- Extensions: remove commutativity, finiteness?

Concise definition

Definition

An **involutive residuated lattice** $\mathbf{A} = \langle A, \vee, \cdot, \sim, -, 0\rangle$ is

- a semilattice $\langle A, \vee\rangle$ and
- a semigroup $\langle A, \cdot\rangle$ such that

$$x \le y \iff x \cdot {\sim}y \le 0 \iff -y \cdot x \le 0 \qquad \text{for all } x, y \in A$$

where $x \le y \iff x \vee y = y$.

It follows that ${\sim}{-}x = x = -{\sim}x$ and $1 = -0$ is an identity for $\cdot$

- $\mathbf{A}$ is **commutative** if $x \cdot y = y \cdot x$ for all $x, y \in A$
- $\mathbf{A}$ is **idempotent** if $x \cdot x = x$ for all $x \in A$

Meet is definable: $x \wedge y = -({\sim}x \vee {\sim}y)$, and commutativity gives ${\sim}x = -x$

More general definition

Definition

A **(pointed) residuated lattice** $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ is

- a lattice $\langle A, \wedge, \vee\rangle$ and
- a monoid $\langle A, \cdot, 1\rangle$ such that

$$x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z \qquad \text{for all } x, y, z \in A.$$

$\mathbf{A}$ is **involutive** if ${\sim}{-}x = x = -{\sim}x$, where ${\sim}x = x\backslash 0$ and $-x = 0/x$.

The concise definition is equivalent to this one via $x\backslash y = {\sim}(-y \cdot x)$ and $x/y = -(y \cdot {\sim}x)$.

CIdInRL denotes the variety of commutative idempotent involutive residuated lattices.

Some properties

Let $\mathbf{A} \in$ CIdInRL.

- $\langle A, \cdot, 1\rangle$ is a meet-semilattice with top element 1 and order $\sqsubseteq$ (**monoidal order**) defined as

$$a \sqsubseteq b \iff a \cdot b = a.$$

Hence, the orders $\leq$ and $\sqsubseteq$ together with the involution $-$ completely determine $\mathbf{A}$, allowing us to work in the signature $\langle A, \vee, \cdot, -, 0, 1\rangle$

Examples I

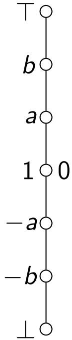
- **Boolean algebras** (where $\leq = \sqsubseteq$)
- **Sugihara monoids** (algebraic semantics for relevance logic RM^{t})

  They are defined as **distributive** commutative idempotent involutive residuated lattices.

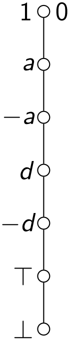
  Dunn [1970] proved that the subdirectly irreducible members in this variety are linearly ordered.

  Up to isomorphism, there is one such algebra $\mathbf{S}_n$ for each chain with n elements.

Examples I



$\langle \mathbf{S}_7, \leq \rangle$



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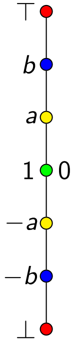

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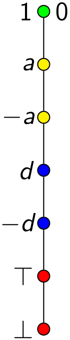

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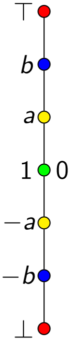

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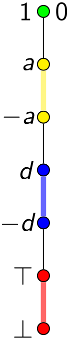

$\langle \mathbf{S}_7, \sqsubseteq \rangle$

Examples II

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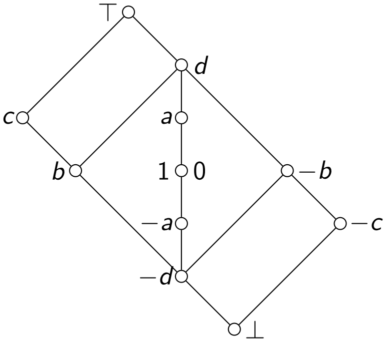
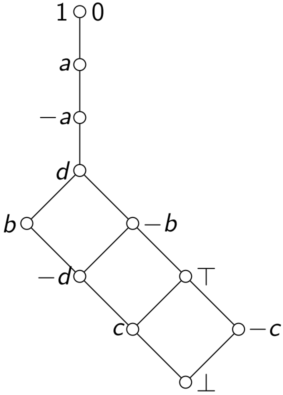
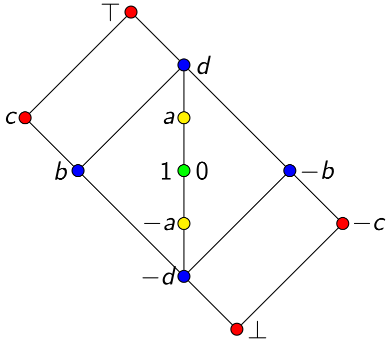


$\langle A, \leq \rangle$



$\langle A, \sqsubseteq \rangle$

Examples II



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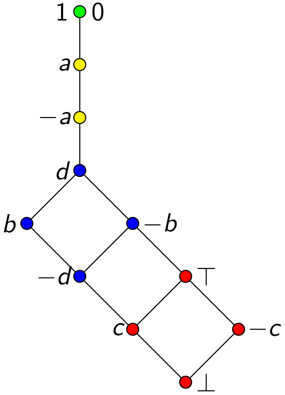

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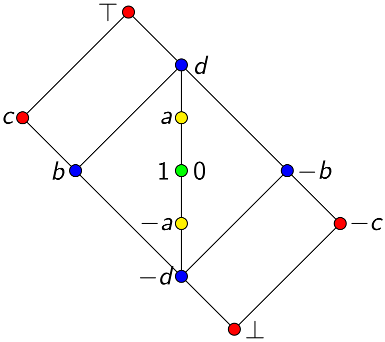

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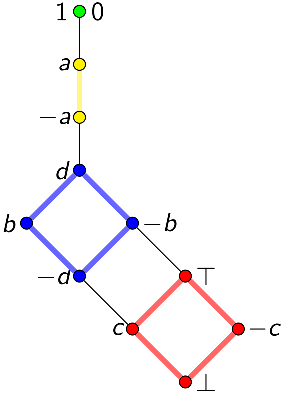


$\langle A, \sqsubseteq \rangle$

Some more properties

For each $x \in A$, let

$$0_x := x \wedge -x = x \cdot -x$$
$$1_x := x \vee -x = -(x \cdot -x) = x/x$$
$$\mathbb{B}_x := \{y \in A \mid 0_x \sqsubseteq y \sqsubseteq 1_x\}$$
$$\downarrow 0 := \{y \in A \mid y \leq 0\} = \{0_x \mid x \in A\}$$

Lemma

- *For each $x \in A$, $\langle \mathbb{B}_x, \wedge, \vee, -, 0_x, 1_x \rangle$ is a* **Boolean algebra**
- *For each $x \in A$, the monoidal order and the lattice order* **agree** *on $\mathbb{B}_x$*
- *The monoidal intervals $\mathbb{B}_x$* **partition** *A*
- $\langle \downarrow 0, \cdot, \vee \rangle$ *is a* **distributive lattice** *with top element* 0

Hence, the monoidal semilattice is a disjoint union of Boolean algebras over the 'skeleton' of a distributive lattice.

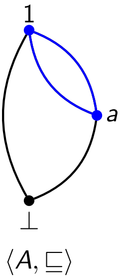
Even more properties

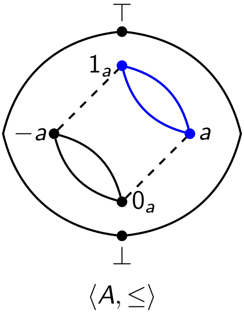
Lemma

Let $\mathbf{A} \in \mathsf{CIdInRL}$ *and* $a \in A$ *such that* $a \leq 1$. *For* $x \in A$,

$$a \sqsubseteq x \iff a \leq x \leq 1_a.$$

Moreover, $\{x \in A \mid a \sqsubseteq x\} = \{x \in A \mid a \leq x \leq 1_a\}$ *is a subuniverse exactly when* $a \leq 0$.





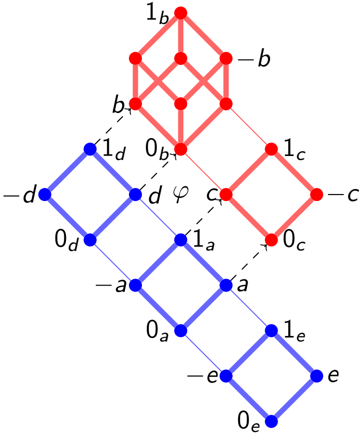
Construction: idea

Goal: Give a structural characterization of all finite members of CIdInRL.

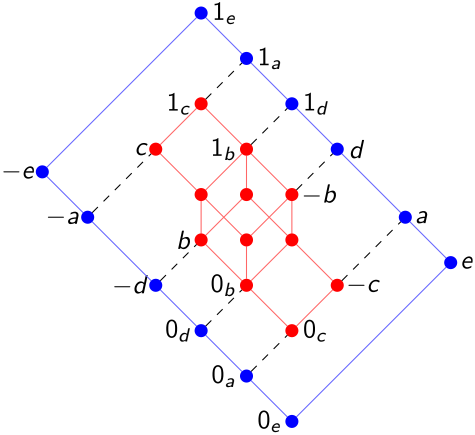
Construction idea: "Consider two algebra $\mathbf{A}, \mathbf{B} \in$ CIdInRL that are 'compatible'.

Construct a new member of the variety **C** by **gluing** the monoidal semilattice of **B** **on top** of that of **A** and the lattice of **B** **in the middle** of the lattice of **A**."

Construction: example



$\langle \mathbf{C}, \sqsubseteq \rangle$



$\langle \mathbf{C}, \leq \rangle$

Construction: formally

$\mathbf{A} = \langle A, \vee^A, \cdot^A, -^A, 0^A, 1^A \rangle$ (the bottom algebra) and
$\mathbf{B} = \langle B, \vee^B, \cdot^B, -^B, 0^B, 1^B \rangle$ (the top algebra) are **φ-compatible** if

- φ is a **bijection** $\Uparrow a \to \Downarrow b$ for some $a \leq 1^A$ and $0^B \leq b \leq 1^B$ such that
- φ **preserves join**, i.e. $\varphi(x \vee^A y) = \varphi(x) \vee^B \varphi(y)$
- φ **preserves fusion**, i.e. $\varphi(x \cdot^A y) = \varphi(x) \cdot^B \varphi(y)$ and
- $0^B = \varphi(a \vee^A 0^A)$.

For φ-compatible algebras we define a **glueing construction** $\oplus_\varphi$

Glueing construction

$$\mathbf{A} \oplus_\varphi \mathbf{B} \coloneqq \langle A \uplus B, \vee, \cdot, -, 1^{\mathbf{B}}, 0^{\mathbf{B}} \rangle$$

$$x \vee y = \begin{cases} x \vee^A y & x, y \in A \\ x \vee^B y & x, y \in B \\ \varphi(x \vee^A a) \vee^B y & x \in A,\ y \in B,\ x \leq^A -^A a \\ x \vee^A \varphi^{-1}(y \cdot^B b) & x \in A,\ y \in B,\ x \not\leq^A -^A a \end{cases}$$

$$x \cdot y = \begin{cases} x \cdot^A y & x, y \in A \\ x \cdot^B y & x, y \in B \\ x \cdot^A \varphi^{-1}(y \cdot^B b) & x \in A,\ y \in B \end{cases}$$

$$-x = \begin{cases} -^A x & x \in A \\ -^B x & x \in B \end{cases}$$

Glueing construction

Theorem

For φ-**compatible** $\mathbf{A}, \mathbf{B} \in \mathsf{CIdInRL}$ *the algebra* $\mathbf{A} \oplus_{\varphi} \mathbf{B}$ *is in* CIdInRL.

The proof is by case analysis and direct computation.

Unglueing decomposition

For finite $\mathbf{C} \in \mathsf{CIdInRL}$, consider a co-atom c in the underlying distributive lattice with universe $\downarrow 0 = \{0_x \mid x \in C\}$.

By distributivity, there exists c^* such that $\langle c, c^* \rangle$ is a splitting pair of $\downarrow 0$.

Note: $c = 0_c$, hence $-c = 1_c$.

Lemma

The pair $\langle 1_c, c^* \rangle$ is a splitting pair of C (in the monoidal order).

Moreover, $\Uparrow c^*$ is a subuniverse of $\mathbf{C}$, and $\Downarrow 1_c$ is closed under $\vee, \cdot, -$

Unglueing decomposition

Let $\mathbf{A} = \langle \Downarrow 1_c, \vee, \cdot, -, 1_c, 0_c \rangle$.

Let $\mathbf{B}$ be the subalgebra of $\mathbf{C}$ with subuniverse $\Uparrow c^*$.

Choose $a = 1_c \cdot c^*$ and $b = (1_c \vee -a) \vee c^*$, and define

$\varphi(x) = (x \wedge -a) \vee c^*$ for $a \sqsubseteq x \sqsubseteq 1_c$.

Lemma

- $a \leq 1_c$ *and* $0 \leq b \leq 1$
- φ *is a bijection to* $\{y \mid c^* \sqsubseteq y \sqsubseteq b\}$ *with* $\varphi^{-1}(y) = y \cdot 1_c$
- $\varphi(c \vee a) = 0_b$

Theorem

The algebra $\mathbf{C} \in \mathrm{CIdInRL}$ *is isomorphic to* $\mathbf{A} \oplus_\varphi \mathbf{B}$.

Structural characterization

Theorem

Any finite member **A** *of* CIdInRL *can be constructed using the gluing construction, starting from finite Boolean algebras.*

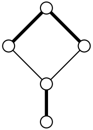
Corollary

Any finite $\mathbf{A} \in$ CIdInRL *is determined by its fusion semilattice and also by its lattice reduct.*

To do: An algorithm for constructing all CIdInRLs

Some examples

Fusion semilattices that **cannot** support a CIdInRL:



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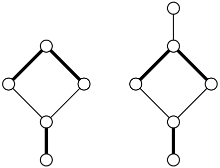

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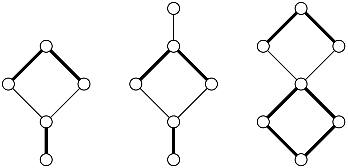

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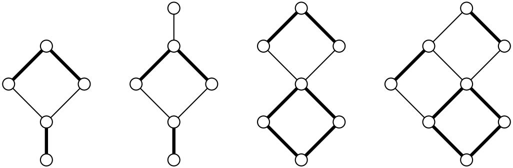

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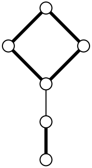

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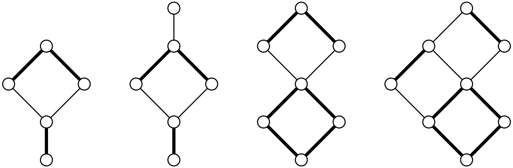

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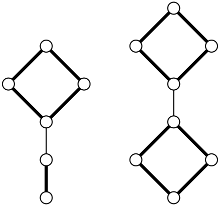

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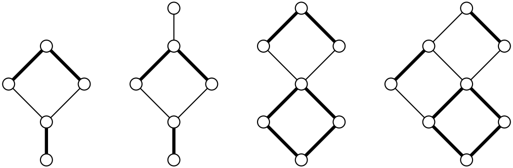

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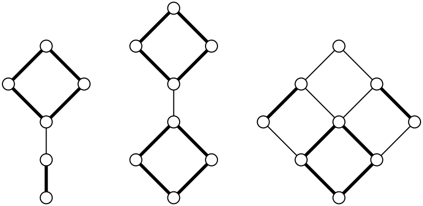
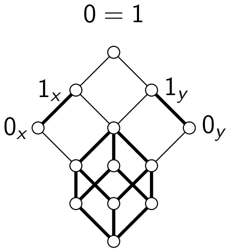


Two more examples and a question for the audience



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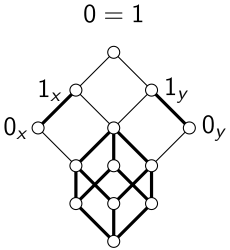
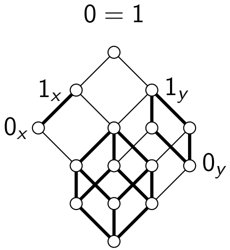




Which one can occur as the fusion semilattice of a CIdInRL?

Fusion-distributivity

As an application, call an $\mathbf{A} \in \mathsf{CIdInRL}$ fusion-distributive if the meet-semilattice $\langle A, \cdot \rangle$ is distributive, i.e. if for all $x, y, z \in A$,

$$x \cdot y \sqsubseteq z \implies \exists x', y' \in A \text{ such that } x \sqsubseteq x', y \sqsubseteq y', \text{ and } z = x' \cdot y'.$$

Lemma

For compatible fusion-distributive $\mathbf{A}, \mathbf{B} \in \mathsf{CIdInRL}$, their gluing $\mathbf{C}$ is fusion-distributive.

Corollary

Any finite $\mathbf{A} \in \mathsf{CIdInRL}$ is fusion-distributive.

Every finite distributive lattice can occur as skeleton

Let D be a finite distributive lattice of height n. Then D can be embedded in the Boolean algebra $\mathbf{2}^n$.

Let $\mathbf{S}_4$ be the 4-element Sugihara monoid. The fusion semilattice is the ordinal sum $\mathbf{2} \oplus \mathbf{2}$.

$(\mathbf{S}_4)^n$ is a glueing of 2^n copies of $\mathbf{2}^n$ over the distributive lattice $\mathbf{2}^n$.

This fusion semilattice contains a sublattice that is the glueing of $|D|$ copies of $\mathbf{2}^n$ over the distributive lattice D.

Locally finite subvarieties

CIdInRL contains many subvarities:

BA $=$ Boolean algebras $=$ variety generated by 2-element BA

$\mathbf{SC}_n$ $=$ variety generated by a Sugihara chain of length n

OddSugi $=$ variety generated by a countable Sugihara chain with $0 = 1$

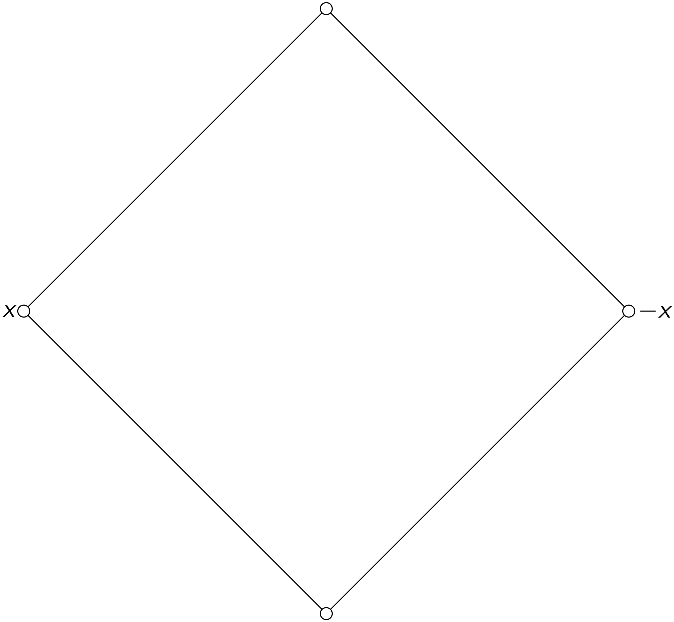
Sugi $=$ variety generated by a countable Sugihara chain with $0 \neq 1$

$\mathbf{V}(A)$ for any finite commutative idempotent involutive residuated lattice

All these varieties are **locally finite**

Is **CIdInRL** locally finite?

A one-generated infinite CIdInRL



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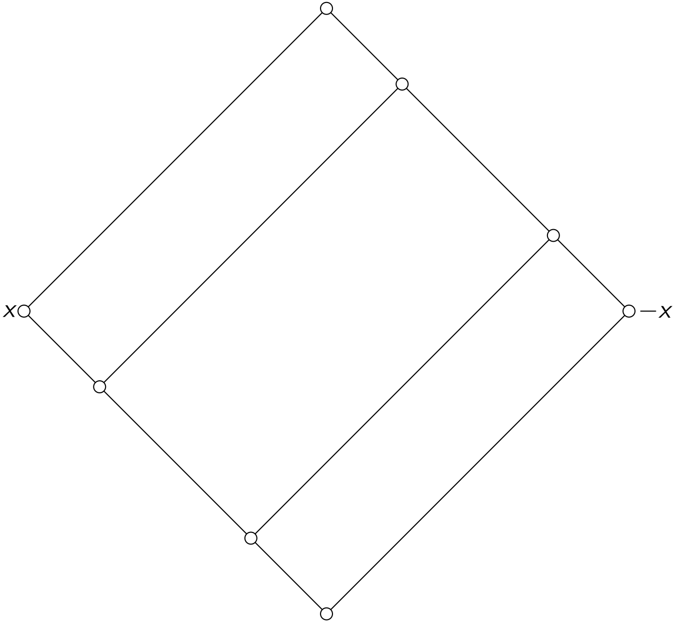

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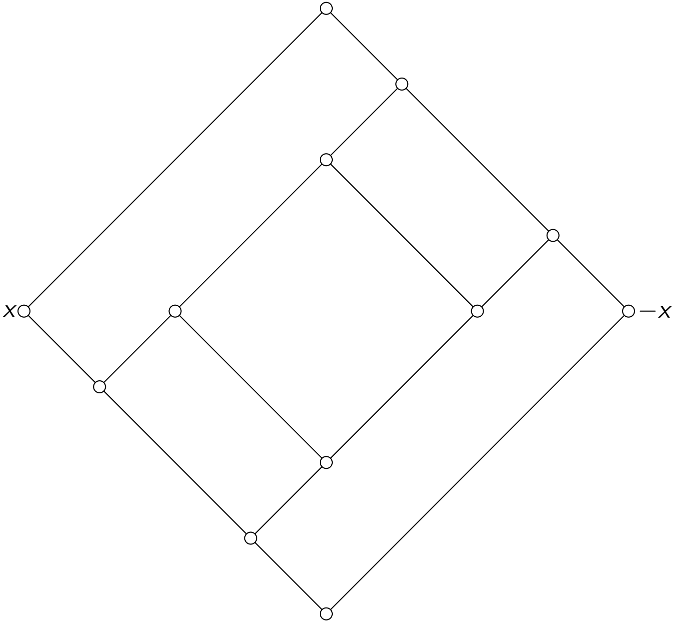

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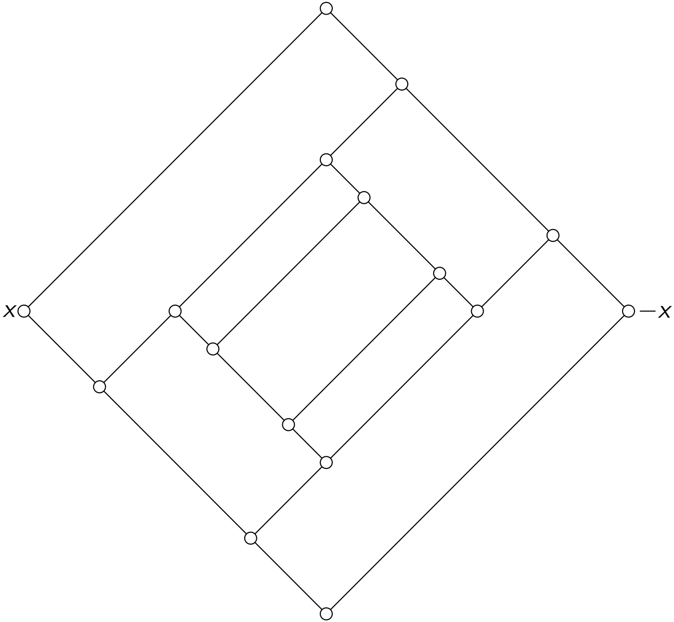

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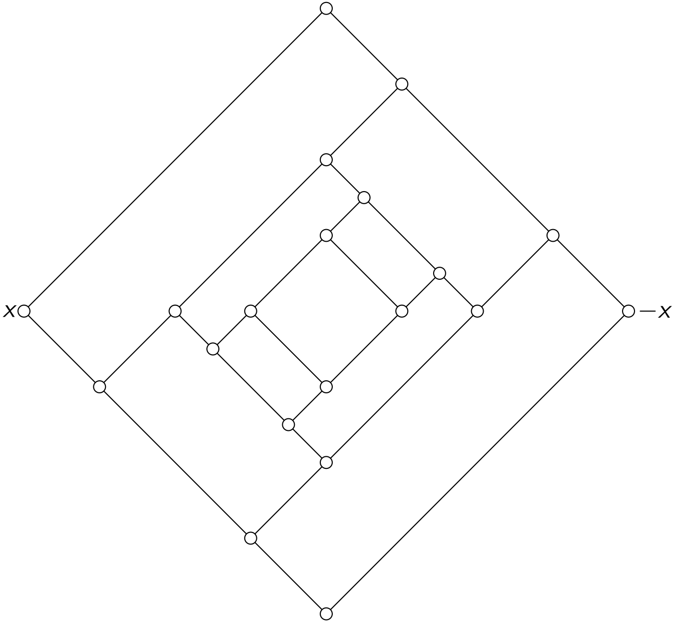

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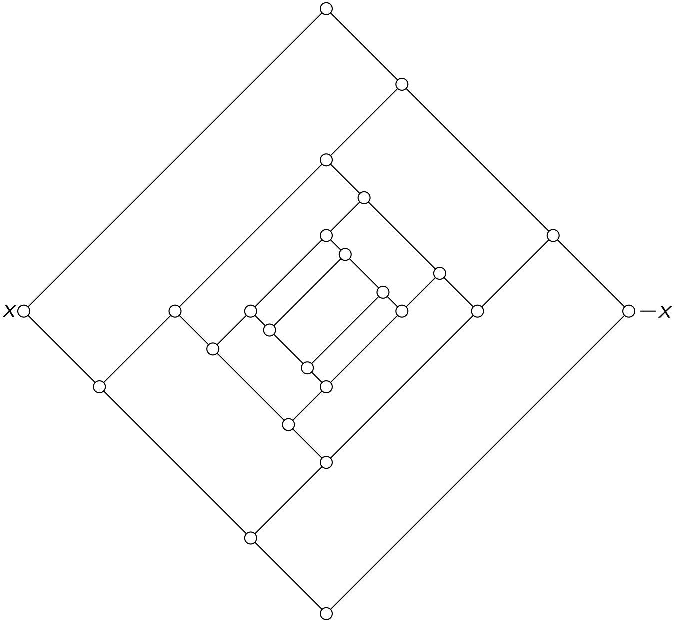

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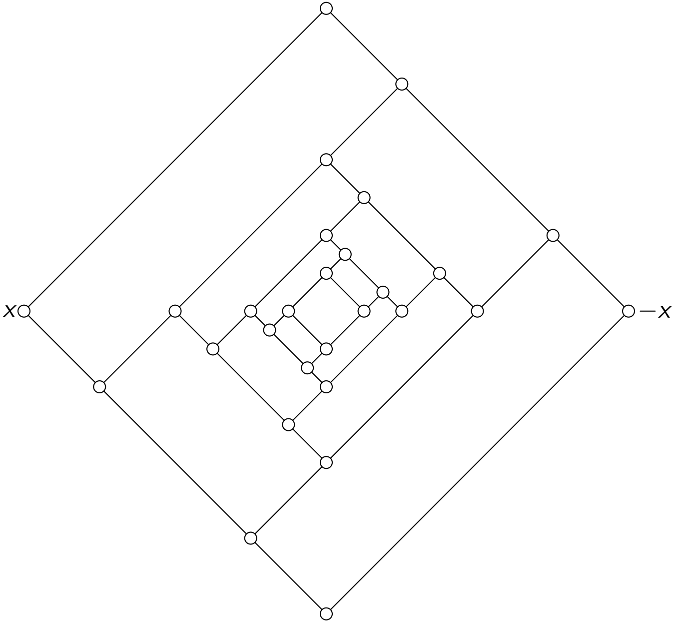

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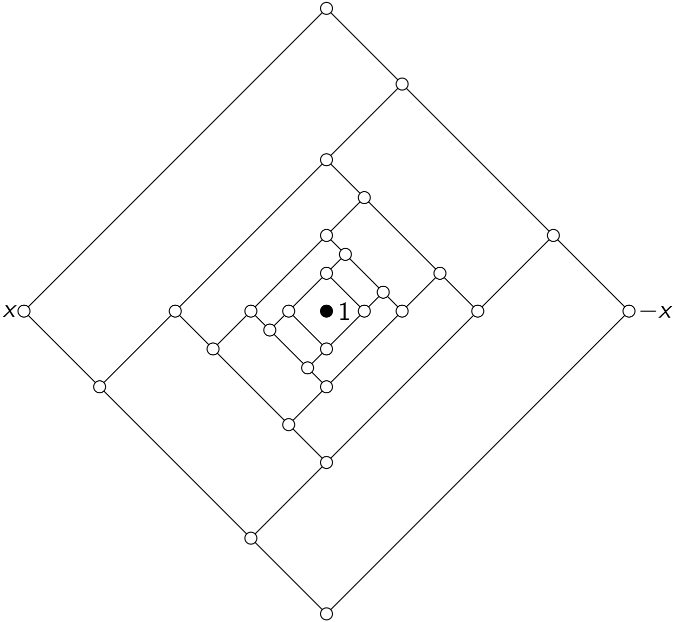

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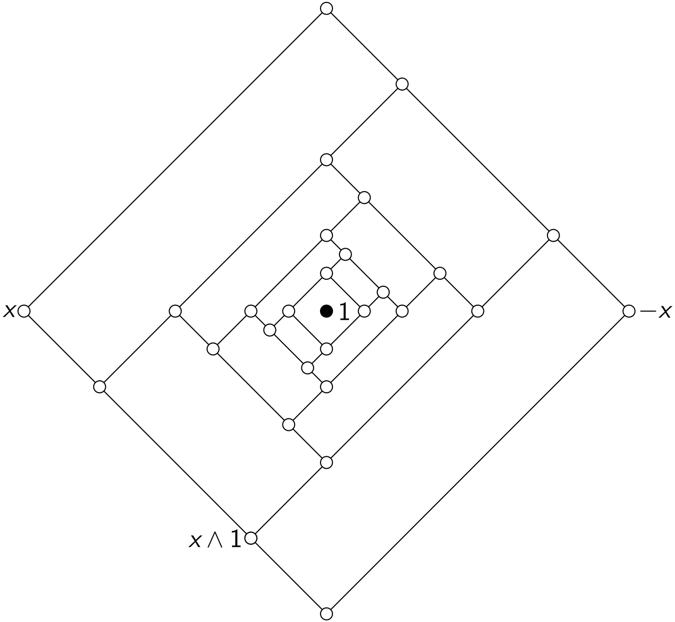

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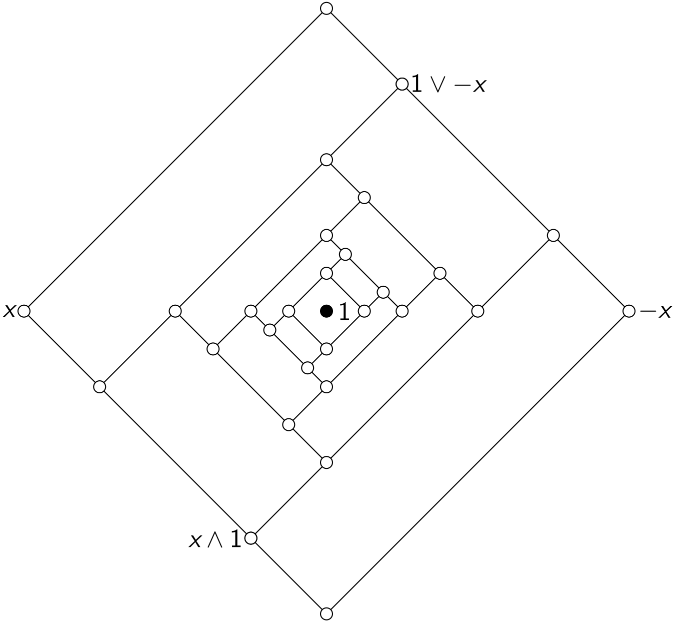

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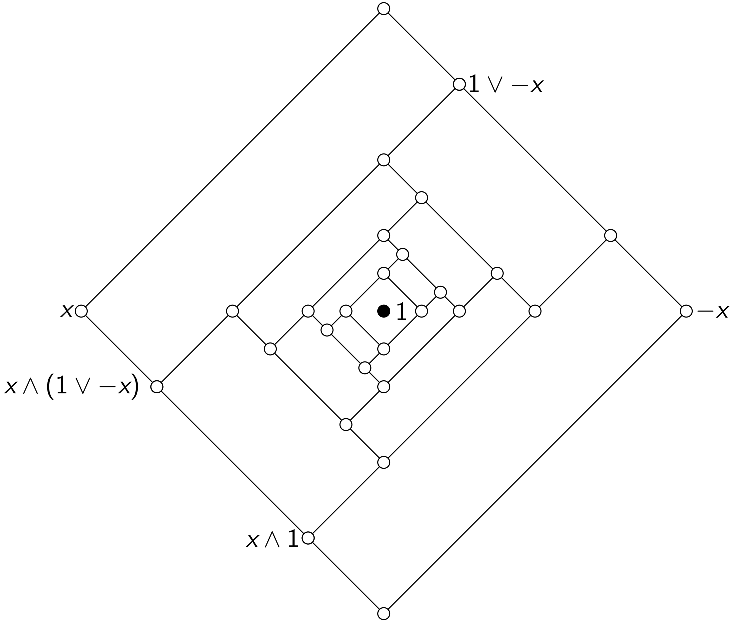

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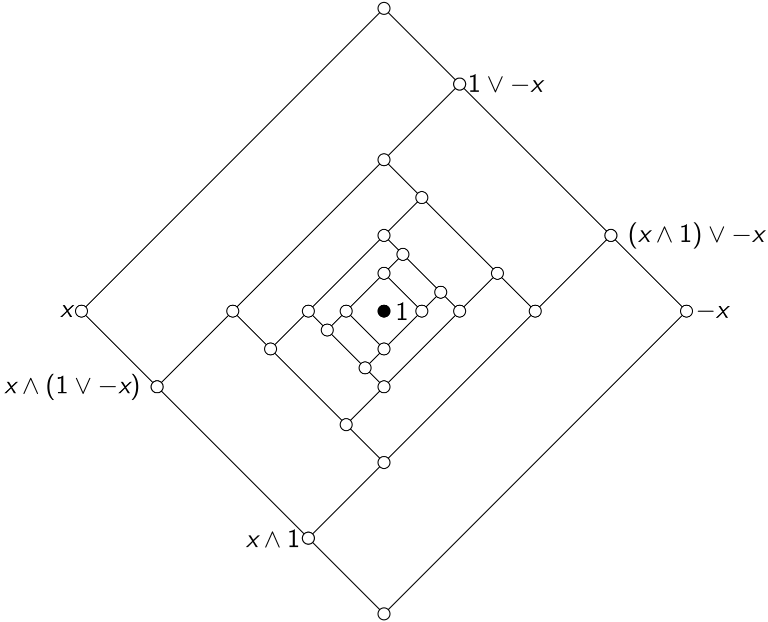
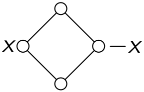


The fusion semilattice of a one-generated infinite CIdInRL



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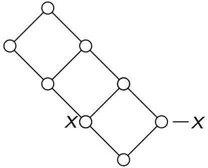

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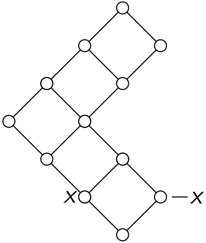
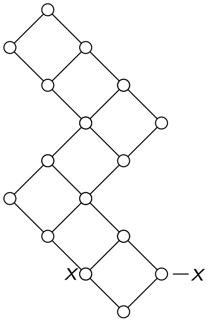
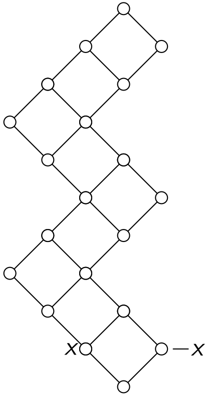


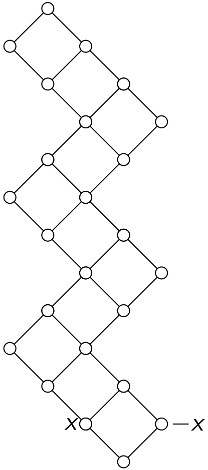
The fusion semilattice of a one-generated infinite CIdInRL



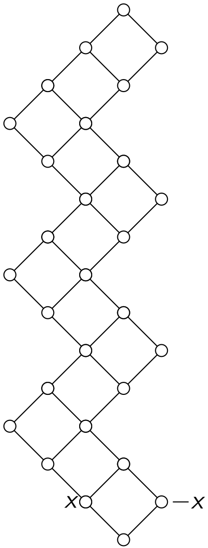
The fusion semilattice of a one-generated infinite CIdInRL



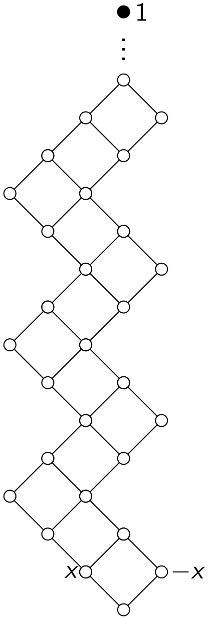
The fusion semilattice of a one-generated infinite CIdInRL



The fusion semilattice of a one-generated infinite CIdInRL



The fusion semilattice of a one-generated infinite CIdInRL



Extensions

- How far can we extend this structural characterization of this variety? From the reverse construction we obtain a structural characterization for all members **A** of CIdInRL for which $\downarrow 0 = \{0_x \mid x \in A\}$ is finite. Can we push this further?
- How much of the results can be generalized to idempotent involutive residuated posets?
- All finite idempotent involutive residuated lattices with $\leq$ **17** elements are known to be **commutative**. Is this true for **all finite** ones?
- Can the glueing construction be used for (some subclass) of non-idempotent (involutive) residuated lattices?
- Can we apply this construction to obtain amalgamation for finite V-formations in CIdInRL?

Towards removing commutativity

An involutive residuated lattice is **cyclic** if $\sim x = -x$

Idempotence for cyclic involutive residuated posets is a **strong restriction**.

Lemma (José Gil-Ferez and PJ)

Any involutive idempotent residuated posets satisfies:

1. $x(\sim x) \leq \sim x$ *and* $(-x)x \leq -x$,
2. $x(\sim x) \leq x$ *and* $(-x)x \leq x$.

Assuming ***cyclicity*** *implies the following additional identities:*

3. $x(\sim x)x = x(\sim x)$,
4. $x(\sim x) = (\sim x)x$.

Proof.

In any involutive residuated poset $\sim(yx) \le \sim(yx)$, so $yx(\sim(yx)) \le 0$, whence $x(\sim(yx)) \le \sim y$.

1. Follows from this identity and idempotence by substituting x for y.
2. Replace x by $\sim x$ in the second identity of (1).
3. Multiplying (1) by x on the right we obtain $x(\sim x)x \le (\sim x)x$. By cyclicity $(\sim x)x \le 0$, and using idempotence gives $xx(\sim x)x \le 0$, or equivalently $x(\sim x)x \le \sim x$. Multiplying by x on the left shows that $x(\sim x)x \le x(\sim x)$. Multiplying (2) by $x(\sim x)$ on the left produces $x(\sim x)x(\sim x) \le x(\sim x)x$, whence $x(\sim x) \le x(\sim x)x$ follows from idempotence. Therefore (3) holds.
4. Again multiplying (1) by x on the right we obtain $x(\sim x)x \le (\sim x)x$, hence by (3) we get $x(\sim x) \le (\sim x)x$. Using cyclicity we can replace x by $\sim x$ to deduce the reverse inequality.

□

Every cyclic idempotent involutive poset is commutative

Theorem (José Gil-Ferez and PJ)

Every ***cyclic idempotent*** *involutive residuated poset is* ***commutative****.*

Proof.

The identity $y \cdot {\sim}(xy) \leq {\sim}x$ holds in any InRL, hence

$$xy \cdot {\sim}(xy) \leq x \cdot {\sim}x \leq {\sim}x.$$

Applying (4) of the preceeding lemma on the left, we have ${\sim}(xy)xy \leq {\sim}x$, from which we deduce ${\sim}(xy)xyx \leq ({\sim}x)x \leq 0$. Therefore $xyx \leq xy$.

Now multiply both sides by y on the left and use idempotence to deduce the identity $yx \leq yxy$. Renaming variables proves $xyx = xy$.

A similar argument shows $xyx = yx$, whence $xy = xyx = yx$. □

A noncyclic idempotent involutive residuated lattice

There exist **noncommutative** idempotent involutive residuated lattices:

Example (Jóse Gil-Ferez and PJ)

Let $A = \mathbb{Z} \oplus \{\mathbf{1}\} \oplus \mathbb{Z}^\partial$, where $\oplus$ is the ordinal sum.

Lattice order:

$\cdots a_{-2} < a_{-1} < a_0 < a_1 < a_2 \cdots < \mathbf{1} < \cdots b_2 < b_1 < b_0 < b_{-1} < b_{-2} \cdots$

Monoid preorder:

$\cdots a_{-2} \equiv b_{-2} \sqsubset a_{-1} \equiv b_{-1} \sqsubset a_0 \equiv b_0 \sqsubset a_1 \equiv b_1 \sqsubset a_2 \equiv b_2 \sqsubset \cdots \sqsubset \mathbf{1}$

Linear negations:

$\mathbf{1} = \mathbf{0}$, $\sim a_i = b_i$, $\sim b_i = a_{i-1}$, $-a_i = b_{i+1}$, $-b_i = a_i$

Hence $\sim\sim a_i = a_{i-1}$ and $--a_i = a_{i+1}$ and the same for b_i.

Conjecture: All **finite** idempotent involutive res. posets are **commutative**.

Some partial results

Theorem

Finite idempotent involutive residuated chains are commutative.

The following results have been obtained using Prover9 [McCune]

Theorem

1. *The po-subvariety of IdInRP determined by the identity* $----x = x$ *satisfies* $--x = x$, *hence is cyclic and thus commutative.*
2. *The po-subvariety of IdInRP determined by the identity* $-------x = x$ *satisfies* $-----x = x$.

Let $-_n x$ be the term with n copies of $-$. Then $-_n x$ is a permutation on A, hence if A is **finite** it satisfies $-_n x = -_m x$ for some $n > m \geq 0$. Applying m copies of $\sim$ on both sides shows A satisfies $-_{n-m} x = x$.

References

[1] K. Blount and C. Tsinakis: The structure of residuated lattices, *Internat. J. Algebra Comput.* **13**, no. 4, 437–461 (2003).

[2] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono, *Residuated Lattices*: *An Algebraic Glimpse at Substructural Logics*, Studies in Logic and the Foundations of Mathematics, vol. 151, Elsevier B. V., 2007.

[3] W. McCune: Prover9 and Mace4 (2010), available at `https://www.cs.unm.edu/~mccune/mace4/`.

Thank you!