From Residuated Lattices via GBI-algebras to BAOs

Peter Jipsen

School of Computational Sciences and
Center of Excellence in Computation, Algebra and Topology (CECAT)
Chapman University

July 22, 2015
22nd Workshop on Logic, Language and Information

Outline

- Nonclassical propositional logics and residuated lattices
- Expansions of residuated lattices
- Generalized bunched implication algebras
- Residuated Boolean monoids
- Proof theory and residuated frames
- Computing finite residuated lattices and GBI-algebras

Nonclassical propositional logics

Classical propositional logic corresponds to **Boolean algebras**

For many applications, **classical logic** is **unnecessarily strong**

Intuitionistic propositional logic does not derive $\varphi \vee \neg\varphi$

Good for **algorithmic reasoning** and **type theory**

Intuitionistic logic corresponds to **Heyting algebras**

Relevance logic does not derive $\psi \to (\varphi \to \psi)$

Considers $\varphi \to \psi$ true only if φ is used in the derivation of ψ

Substructural logic generalizes **many** such weaker logics

It uses a (possibly) **noncommutative dynamic conjunction** (**fusion**), denoted $\cdot$, which is associative but lacks some of the structural laws, e.g., **contraction** $\frac{\varphi\cdot\varphi\Rightarrow\psi}{\varphi\Rightarrow\psi}$ or **weakening** $\frac{\varphi\Rightarrow\psi}{\varphi\cdot\theta\Rightarrow\psi}$

Substructural logics – Residuated lattices

Substructural logics correspond to **residuated lattices**

A **residuated lattice** $(A, \vee, \wedge, \cdot, 1, \backslash, /)$ is an algebra where $(A, \vee, \wedge)$ is a **lattice**, $(A, \cdot, 1)$ is a **monoid** and for all $x, y, z \in A$

$$x \cdot y \leq z \iff y \leq x\backslash z \iff x \leq z/y$$

FL = Full Lambek calculus = the starting point for **substructural logics**

An **FL-algebra** is a residuated lattices **with a new constant** 0

Extensions of substructural logic correspond to **subvarieties** of FL-algebras

Residuated lattices and **FL-algebras** generalize many algebras related to **logic**, e. g. **Boolean algebras**, **Heyting algebras**, **MV-algebras**, **Gödel algebras**, **Product algebras**, Hajek's **basic logic algebras**, **linear logic algebras**, **lattice-ordered (pre)groups**, ...

Hiroakira Ono

(California, September 2006)

[1985] *Logics without the contraction rule*

(with Y. Komori)

Provides a **framework** for studying many substructural logics, relating sequent calculi with semantics

The name **substructural logics** was suggested by K. Dozen, October 1990

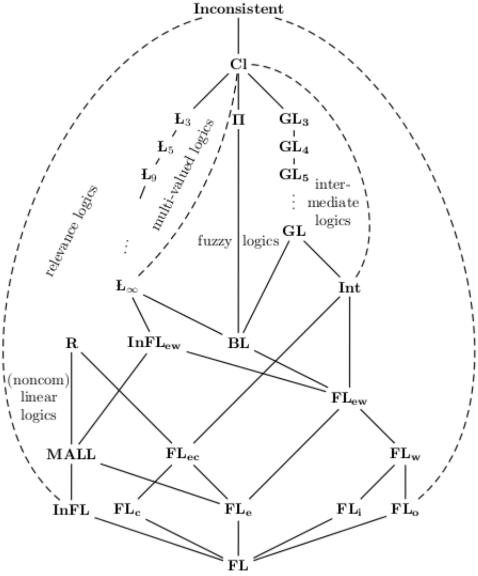
[2007] *Residuated Lattices: An algebraic glimpse at substructural logics* (with Galatos, J., Kowalski)

Some axioms for subclasses of RL

Logic	**Algebra**	**Axioms**	**w/o** 0
Full Lambek Calculus	FL-algebras	Lattice+Mon+$\backslash, /, 0$	RL
Intuition. Linear Logic	FL_e-algebras	FL + $xy{=}yx$	CRL
FL+exchange+weak.	FL_{ew}-algebras	FL_e+ $0{\wedge}x{=}0, 1{\vee}x{=}1$	CIRL
Intuitionistic Logic	**Heyting algs**	FL_{ew} + $x{\wedge}y{=}xy$	**GHA**
Classical Logic	Boolean algebra	HA + $\neg\neg x{=}x$	GBA

Most results proved for residuated lattices apply to **all** subclasses

Some propositional logics extending FL



Algebraic terms = propositional formulas

Residuated lattices form an equational class:

$$\begin{array}{ccc} (x \vee y) \vee z = x \vee (y \vee z) & (xy)z = x(yz) & x(x\backslash z \wedge y) \vee z = z \\ (x \wedge y) \wedge z = x \wedge (y \wedge z) & x1 = x = x1 & x\backslash(xz \vee y) \wedge z = z \\ x \vee y = y \vee x & x \vee (x \wedge y) = x & (y \wedge z/x)x \vee z = z \\ x \wedge y = y \wedge x & x \wedge (x \vee y) = x & (y \vee zx)/x \wedge z = z \end{array}$$

Define $x \leq y$ if and only if $x \wedge y = x$

$\varphi \vdash \psi$ holds in **substructural logic** iff $\varphi \leq \psi$ is valid in all **residuated lattices**

In particular, $\vdash \psi$ is a **theorem** iff $1 \leq \psi$ is valid in all residuated lattices

So we can use **Birkhoff's equational logic** to understand substructural logics

More importantly, we have **algebraic semantics** for **counterexamples**

Homomorphic images of Residuated Lattices

A map $h : A \to B$ is a **homomorphism** if $h(x \diamond y) = h(x) \diamond h(y)$ for $\diamond \in \{\vee, \wedge, \cdot, \backslash, /\}$ and $h(1) = 1$

If h is **surjective**, we say that B is a **homomorphic image** of A

Recall that for groups the homomorphic images are (up to isomorphism) in 1-1 correspondence with **normal subgroups** of the domain

This is **not true** for lattices or monoids, so the next result is **interesting**:

Theorem

[Blount, Tsinakis 2003] Homomorphic images of residuated lattices *are determined by* ***convex normal subalgebras****.*

A subset N is **convex** if $x, y \in N$ and $x \leq z \leq y$ imply $z \in N$

N is **normal** if for all $a \in A$ and $x \in N$, $a\backslash xa \wedge 1$ and $ax/a \wedge 1$ are in N

N is a **subalgebra** if it is closed under the operations $\{\vee, \wedge, \cdot, 1, \backslash, /\}$

Structure of finite residuated lattices

Call an element $e \in A$ a **negative central idempotent** if $ee = e \leq 1$ and $ex = xe$ for all $x \in A$

Then an example of a **convex normal subalgebra** is the interval $[e, 1/e]$

Theorem

*In a **finite** residuated lattice this describes **all** convex normal subalgebras*

An equivalence relation $\theta \subseteq A^2$ is a **congruence** if

$$(a, b), (c, d) \in \theta \text{ implies } (a \diamond c, b \diamond d) \in \theta \text{ for all } \diamond \in \{\vee, \wedge, \cdot, 1, \backslash, /\}$$

The set of all congruences forms a **complete lattice under subset-inclusion** (in any universal algebra)

Theorem

[Galatos 03] Let A be finite and let $C(A)$ be its set of negative central idempotents. Then $C(A)$ with the induced order of A is a distributive lattice and is dually isomorphic to the congruence lattice of A.

Expansions of Residuated Lattices

Can add an unlimited number of operations

In practice: $0, \bot, \top, !, ?, ^*, \Diamond, \Box, +, \to$

Adding 0 is most common, producing **FL-algebras**

$\implies$ **linear negations**: $\sim x = 0\backslash x \quad$ and $\quad -x = x/0$

Involutive FL-algebras are defined by $\sim - x = x = - \sim x$

Cyclic FL-algebras are defined by $\sim x = -x$
Add commutativity and exponentials $!, ?$ to get **linear logic**
Add * to FL_o to get **residuated Kleene lattices**
Add $\Diamond, \Box$ to FL to get **modal FL-algebras**

All of these expansions are examples of **Lattices with Operators**
I.e., lattices with operations that are **order-preserving or order-reversing** in each argument

Lattices with operators and subclasses

LO	$\supseteq$	Residuated lattices
|		|
HAO	$\supseteq$	Generalized bunched implication algebras
|		|
BAO	$\supseteq$	Residuated Boolean monoids

Generalized bunched implication algebras

Recall that a **Heyting algebra** is an FL-algebra with $0 = \bot$ as bottom element and $xy = x \wedge y$

In this case we write $x \to y$ instead of $x \backslash y \quad (= y/x)$

Also define $\neg x = x \to \bot$ and $\top = \neg \bot$

A **generalized bunched implication algebra** or **GBI-algebra** is an algebra $(A, \vee, \wedge, \to, \bot, \cdot, 1, \backslash, /)$ where $(A, \vee, \wedge, \to, \bot)$ is a **Heyting algebra**, and $(A, \vee, \wedge, \cdot, 1, \backslash, /)$ is a **residuated lattice**

Theorem

The equational theory of GBI-algebras is ***decidable***

BI-algebras are commutative GBI-algebras

Applications in computer science; basis of **separation logic**

Another example: Heyting relation algebras

A **Heyting relation algebra** has the form $(A, \vee, \wedge, \to, \bot, ;, 1, \backslash, /, \sim)$ where $(A, \vee, \wedge, \to, \bot)$ is a **Heyting algebra** and $(A, \vee, \wedge, \to, \bot, ;, 1, \backslash, /, \sim)$ is a **cyclic involutive residuated lattice**

Hence $(A, \vee, \wedge, \to, \bot, \sim)$ is a **symmetric Heyting algebra** in the sense of **A. Monteiro**

Connection to **relation algebras**: Let $(P, \sqsubseteq)$ be a preorder

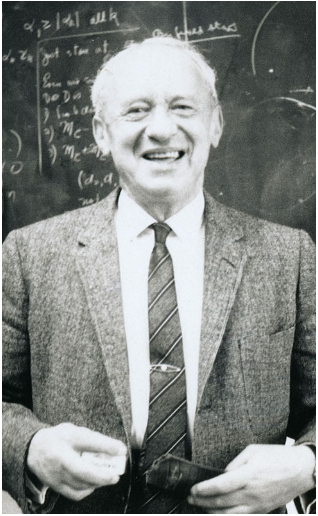
$R \subseteq P^2$ is a **weakening relation** if $\sqsubseteq; R; \sqsubseteq = R$

The set $W(P)$ of **all weaking relations** is closed under $\bigcup, \bigcap, ;$

$\sqsubseteq$ is the **identity element** w.r.t. composition

$\backslash, /$ and $\to$ **exist** since ; and $\cap$ **distribute** over $\bigcup$

Algebraic logic



Alfred Tarski

(May 1967, visiting at U. of Michigan)

According to the MacTutor Archive, **Tarski** is recognised as one of the four greatest logicians of all time, the other three being **Aristotle**, **Frege**, and **Gödel**

Of these **Tarski** was the most prolific as a logician

His collected works, excluding the 20 books, runs to 2500 pages

Boolean algebras with operators

Bjarni Jónsson

(AMS-MAA meeting in Madison, WI 1968)

Boolean Algebras with operators, Part I and Part II [1951/52] with **Alfred Tarski**

One of the cornerstones of algebraic logic

Constructs **canonical extensions** and provides **semantics** for multi-modal logics

Gives representation for **abstract relation algebras** by **atom structures**

Boolean algebras with operators

Let $\tau = \{f_i : i \in I\}$ be a set of operation symbols, each with a fixed finite arity

$\mathbf{BAO}_\tau$ is the class of algebras $(A, \vee, \wedge, \neg, \bot, \top, f_i\,(i \in I))$ such that $(A, \vee, \wedge, \neg, \bot, \top)$ is a **Boolean algebra** and the f_i are **operators** on A

i.e., $f_i(\ldots, x \vee y, \ldots) = f_i(\ldots, x, \ldots) \vee f_i(\ldots, y, \ldots)$

and $f_i(\ldots, \bot, \ldots) = \bot$ for all $i \in I$ (so the f_i are **strict**)

BAOs are the **algebraic semantics** of classical **multimodal logics**

Main result: every BAO $\mathbf{A}$ can be embedded in its **canonical extension** $\mathbf{A}^\sigma$, a **complete and atomic** Boolean algebra with operators

The **set of atoms** of this Boolean algebra is the **Kripke frame** of the multimodal logic

Example: Residuated Boolean monoids

A **residuated Boolean monoid** is an algebra $(A, \vee, \wedge, \neg, \bot, \top, \cdot, 1, \triangleright, \triangleleft)$ such that $(A, \vee, \wedge, \neg, \bot, \top)$ is a **Boolean algebra**, $(A, \cdot, 1)$ is a **monoid** and for all $x, y, z \in A$

$$(x \cdot y) \wedge z = \bot \iff (x \triangleright z) \wedge y = \bot \iff (z \triangleleft y) \wedge x = \bot$$

Rewrite this as

$$x \cdot y \leq z \iff y \leq \neg(x \triangleright \neg z) \iff x \leq \neg(\neg z \triangleleft y)$$

Define $x\backslash z = \neg(x \triangleright \neg z)$ and $z/y = \neg(\neg z \triangleleft y)$, to see that the variety of residuated Boolean monoids is term-equivalent to the variety of Boolean GBI-algebras (i.e., $\neg\neg x = x$ where $\neg x = x \to \bot$)

Theorem

[Jónsson, Tsinakis 1992] ***Relation algebras*** *are a subvariety of residuated Boolean monoids*

$\implies$ **Relation algebras** are (term-equivalent to) $\subseteq$ **Boolean GBI-algebras**

Boolean + associative operator ⇒ undeciable

Theorem

[Tarski 1941] The class of ***representable relation algebras*** *has an* ***undecidable equational theory****, and the same holds for the variety of* ***(abstract) relation algebras***

Theorem

[Andreka, Kurucz, Nemeti, Sain, Simon 95, 96] The equational theories of ***Boolean GBI-algebras*** *(= residuated Boolean monoids) and* ***Boolean BI-algebras*** *(= commutative residuated Boolean monoids), as well as a large interval of other varieties, are* ***undecidable***

Lattices with operators

Gehrke and Harding [2001] develop canonical extensions for **lattices with operators**

Dunn, Gehrke, Palmigiano [2005] define **generalized Kripke frames** using (maximally disjoint) **filter–ideal pairs**

For the lattice reducts, this is based on G. Birkhoff's **polarities**, A. Urquhart's **lattice spaces** and the notion of **contexts** from R. Wille's **Formal Concept Analysis**

Expansions of residuated lattices **by operators** fit into this theory

However, integrating the **proof theory** of residuated lattices and their **reducts/expansions** requires further ideas

A glimpse of algebraic proof theory

Gentzen [1936] defined **sequent calculi**, including **LK** (for classical logic) and **LJ** (for intuitionistic logic)

For **proof search** and **proof normalization**, he proved that the **cut rule** can be **omitted** without affecting provability

Example: A a **sequent calculus** for residuated lattices

Let **RL** be the **equational theory** of **residuated lattices**

Let $T = Fm_{\vee,\wedge,\cdot,1,\backslash,/}(x_1, x_2, \ldots)$, $\quad W = F_{Mon(\circ,\varepsilon)}(T)$, $\quad W' = U \times T$

where $U = \{u \in F_{Mon(\circ,\varepsilon)}(T \cup \{x_0\}) : u$ contains **exactly one** $x_0\}$

The Gentzen system GL

A **Horn formula** $\varphi_1 \& \cdots \& \varphi_n \to \psi$ is written $\frac{\varphi_1 \ \cdots \ \varphi_n}{\psi}$

Let $a, b, c \in T$, $\quad s, t \in W \quad$ and $\quad u \in U$

GL:

$$\frac{}{a \Rightarrow a} \qquad \frac{t \Rightarrow a}{t \Rightarrow a \vee b} \qquad \frac{t \Rightarrow b}{t \Rightarrow a \vee b} \qquad \frac{u(a) \Rightarrow c \quad u(b) \Rightarrow c}{u(a \vee b) \Rightarrow c}$$

$$\frac{t \Rightarrow a \quad u(a) \Rightarrow b}{u(t) \Rightarrow b}(\text{cut}) \qquad \frac{u(a) \Rightarrow c}{u(a \wedge b) \Rightarrow c} \qquad \frac{u(b) \Rightarrow c}{u(a \wedge b) \Rightarrow c} \qquad \frac{t \Rightarrow a \quad t \Rightarrow b}{t \Rightarrow a \wedge b}$$

$$\frac{u(a \circ b) \Rightarrow c}{u(a \cdot b) \Rightarrow c} \qquad \frac{s \Rightarrow a \quad t \Rightarrow b}{s \circ t \Rightarrow a \cdot b} \qquad \frac{}{\varepsilon \Rightarrow 1} \qquad \frac{u(\varepsilon) \Rightarrow a}{u(1) \Rightarrow a}$$

$$\frac{a \cdot t \Rightarrow b}{t \Rightarrow a \backslash b} \qquad \frac{t \Rightarrow a \quad u(b) \Rightarrow c}{u(t \circ (a \backslash b)) \Rightarrow c} \qquad \frac{t \cdot b \Rightarrow a}{t \Rightarrow a / b} \qquad \frac{t \Rightarrow b \quad u(a) \Rightarrow c}{u((a/b) \circ t) \Rightarrow c}$$

Example of a *cut-free* **RL** proof $\dfrac{\dfrac{\dfrac{\dfrac{z \Rightarrow z \quad x \Rightarrow x}{z \circ (z \backslash x) \Rightarrow x}}{z \circ (z \backslash x \wedge z \backslash y) \Rightarrow x} \quad \dfrac{\dfrac{z \Rightarrow z \quad y \Rightarrow y}{z \circ (z \backslash y) \Rightarrow y}}{z \circ (z \backslash x \wedge z \backslash y) \Rightarrow y}}{z \circ (z \backslash x \wedge z \backslash y) \Rightarrow x \wedge y}}{z \backslash x \wedge z \backslash y \Rightarrow z \backslash (x \wedge y)}$

Semantics of sequent calculi: Residuated frames

Let $\mathbf{GL}_{cf}$ be the sequent calculus **GL without** the **cut rule**

Define a binary relation $N \subseteq W \times W'$ by

$$wN(u,a) \iff u(w) \Rightarrow a \quad \text{is provable in } \mathbf{GL}_{cf}$$

Define the **accessibility** relations $R_\circ \subseteq W^3$, $R_{\backslash\backslash}, R_{//}$ by

$$R_\circ(v_1, v_2, w) \iff v_1 \circ v_2 = w$$

$$R_{\backslash\backslash} = \{((u,a), x, (u(\_ \circ x), a)) : u \in U, a \in T, x \in W\}$$

$$R_{//} = \{(x, (u,a), (u(x \circ \_), a)) : u \in U, a \in T, x \in W\}$$

Then $(W, W', N, R_\circ, R_{\backslash\backslash}, R_{//})$ is a **residuated frame**

(A **general** residuated frame is $(W, W', N, R_i(i \in I))$)

Algebraic cut-admissibility

Theorem

[Okada, Terui 1999, Galatos, J. 2013]. The following are equivalent:

1. $t \Rightarrow a$ *is provable in* ***GL***
2. $t \leq a$ *holds in* ***RL***
3. $t \Rightarrow a$ *is provable in* $\boldsymbol{GL}_{cf}$

Proof (outline): (3⇒1) is obvious. (1⇒2) Assume $t \Rightarrow a$ is provable **with cut**. Show that **all sequent rules** hold as quasiequations in **RL** (where $\Rightarrow, \circ$ are **replaced by** $\leq, \cdot$)

(2⇒3) Assume $t \leq a$ holds in **RL** and define an algebra $\mathbf{W}^+ = (C[\mathcal{P}(W)], \cup, \cap, \cdot, 1, \backslash, /)$ using the **closed sets** $C(X)$ of the **polarity** (W, W', N) and

$X \cdot Y = C(\{w : R(v_1, v_2, w)$ for some $v_1 \in X, v_2 \in Y\})$

$X\backslash Y = \{w \in W : X \cdot \{w\} \subseteq Y\}$ $\qquad Y/X = \{w \in W : \{w\} \cdot X \subseteq Y\}$.

Proof outline (continued)

Then $\mathbf{W}^+$ is a residuated lattice, hence satisfies $t \le a$

Let $f : T \to \mathbf{W}^+$ be a **homomorphism**

Extend to $\bar{f} : W \to \mathbf{W}^+$, so $t \le a$ implies $\bar{f}(t) \subseteq \bar{f}(a)$

Define $\{b\}^{\triangleleft} = \{w \in W : wN(x_0, b)\}$

Prove by **induction** that $b \in \bar{f}(b) \subseteq \{b\}^{\triangleleft}$ for all $b \in T$

Then $t \in \bar{f}(t) \subseteq \bar{f}(a) \subseteq \{a\}^{\triangleleft}$, hence $tN(x_0, a)$

Therefore $\quad t \Rightarrow a \quad$ holds in $\mathbf{GL}_{cf}$ $\square$

Theorem

The equational theory of residuated lattices is ***decidable****. Moreover,* **RL** *has the* ***finite model property***

[Galatos, J. 2013] The variety of integral RL (i.e., $x \wedge 1 = x$) has the ***finite embedding property****, hence the* ***universal theory is decidable****.*

Expanding this approach to GBI-algebras

A similar approach can be used to prove that the equational theory of GBI-algebras is decidable

Add Gentzen rules for an external connective $\owedge$ corresponding to $\wedge$, and rules for $\to$

Expand the residuated frame with a ternary relation for $\owedge$

Theorem

[Galatos, J.] The equational theory of GBI-algebras is ***decidable****.*
Moreover, (G)BI-algebras have the ***finite model property***

Theorem

[Galatos, J.] The variety of integral GBI-algebras (i.e., $x \wedge 1 = x$) has the ***finite embedding property****, hence the* ***universal theory is decidable****.*

How to compute finite residuated lattices

First compute all **lattices** with n elements (up to isomorphism)

[J. and Lawless 2015]: For $n = \mathbf{19}$ there are **1 901 910 625 578**

Then compute all **lattice-ordered monoids** with **zero** ($\perp$) over each lattice

The residuals are **determined** by the monoid

There are **295292 residuated lattices** of size $n = \mathbf{8}$

[Belohlavek and Vychodil 2010]: For **commutative integral** residuated lattices there are **30 653 419** of size $n = \mathbf{12}$

Demo (?)

Conclusion

Substructural logics and **residuated lattices** are an excellent **framework** for investigating and **comparing** propositional logics

By considering **expansions** many more propositional logics are covered

Between (**D**)**LOs** and **BAOs** there is much **uncharted territory**

The success of **bunched implication logic** and **separation logic** in **program verification** provide justification for **more research** in this area

Algebraic, **semantic** and **proof theoretic** techniques can often be adapted to the **expansions**

Some References

J. M. Dunn, M. Gehrke and A. Palmigiano, Canonical extensions and relational completeness of some substructural logics, J. Symbolic Logic, 70(3) 2005, 713–740

N. Galatos and P. Jipsen, Residuated frames with applications to decidability, Trans. of AMS, 365, 2013, 1219–1249

N. Galatos and P. Jipsen, Distributive residuated frames, preprint

N. Galatos, P. Jipsen, T. Kowalski, H Ono, Residuated Lattices: An Algebraic Glimpse at Substructural Logics. No. 151 in Studies in Logic and the Foundations of Mathematics. Elsevier (2007).

M. Gehrke, Generalized Kripke frames, Studia Logica, 84, 2006, 241–275

M. Gehrke and J. Harding, Bounded lattice expansions, Journal of Algebra, 238, 2001, 345–371

Thank You